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DEPARTMENTS

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*Improving the program in mathematics in Oklahoma schools**

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As elsewhere, improvement in mathematics teaching is taking place in Oklahoma.

THE IMPROVEMENT of the mathematics program in our schools at all levels is long overdue, and the need for it is urgent. An improvement program is definitely underway in Oklahoma schools. Many things are happening in the field of mathematics and mathematics education. From the standpoint of content, mathematics is one of the fastest growing and most radically changing of the sciences. It has been described as "the only branch of learning in which all of the major theories of 2,000 years ago are still valid, yet never before has there been such a flood of new ideas." This involves highly theoretical developments, new branches of mathematics highly useful in modern life as well as new uses for old branches of the subject. In many respects mathematics is an entirely different discipline from what it was in 1900 and from what is being presently taught in most schools.

Though much more could be said about present-day mathematics and its relations to life in the second half of the twentieth century, we shall be more interested in the recognition this development is receiving in the schools and the effect of this in the form of changes that are taking place in the point of view from which an improved program can be taught. Teachers, administrators, and patrons have become concerned about the changing point of view in teaching the subject. Men in high places have offered criticisms and suggestions for reform, "often made in most general terms in short speeches or published in condensed form—sometimes written by reporters unfamiliar with both new and old mathematics." The panacea is often to teach "modern" mathematics; more will be said about this later.

THE TRADITIONAL MATHEMATICS PROGRAM

We do have a traditional mathematics program in the schools. This involves both the content and point of view in teaching it. There has been an amazing conservatism or lack of change in the way mathematics has been taught in the schools. Everyone, whether a teacher or not, is familiar with this from his own experience with arithmetic in the grade school, and with algebra, geometry, and advanced algebra in the high school to the calculus in college. By some curious coincident the field of mathematics, though it has expanded continuously over a period of 5,000 years, became a static, almost a stagnant, subject in the classrooms of this country. Subject matter for mathematics from arithmetic through the calculus in college was crystallized into its present form approximately 60 years ago and has changed little since that time. The subject matter, which was developed for the most part over 300 years ago (in the case of plane and solid geometry it was over 2,000 years ago), was chosen and its presentation organized in accordance with an attitude

* Read on October 1, 1960, to Annual Meeting of Oklahoma School Administrators at Norman, Oklahoma.

toward mathematics which no longer holds and which has been discarded by present-day administrators, teachers, and mathematicians. With this curriculum and point of view mathematics is presented mainly as a collection of slightly related techniques and manipulations. The profound, yet simple, concepts get little attention. It has been said that if art and art appreciation "were taught in the same way, it would consist mostly of how to chip stone and mix paints."

In the area of algebra, as an illustration, we may say that "at the beginning of the nineteenth century, algebra consisted of a set of rules and devices for performing operations on real numbers and symbols representing real numbers (manipulations of algebraic expressions), solutions by radicals of polynomials up to the fourth degree, and approximate solutions of polynomials of any degree. This we think of as classical algebra; it is the algebra presented today in traditional elementary textbooks."[1]

MODERN OR CONTEMPORARY MATHEMATICS PROGRAM

The use of the term "modern mathematics" in connection with proposed curricula changes in the high-school program is unfortunate and often misleading. It implies that a totally different program is being proposed and that it consists of concepts from the field of mathematics that have been developed recently, say the last 50 years. This is not true. The mathematics being proposed in the newer programs presents essentially the same subject matter as before, but it is developed from a different point of view and use is made of a relatively small number of new concepts such as sets, algebraic systems, binary operations, axiomatic systems, etc. We now believe that the subject should be taught and learned with a conscious emphasis on the point of view that mathematics is concerned with abstract patterns of thought or a mathematical structure and that by this means students will acquire an understanding of and an interest in mathematics as a cultural as well as an applied subject.

Hence, a modern treatment of algebra identifies an *algebraic system* which consists of a set of elements (numbers), a set of operations (+, ×, −) defined on the elements, and properties derived from these definitions. By a property of an operation is meant a relationship among the elements and the operation which remains true for *all* the elements of the set. Some properties of the set of real numbers under the operations of addition (+) and multiplication (×) are *closure, commutativity, associativity*, and *distributivity*.

Using the idea of structure and new terms, symbols and concepts, when these will clarify the discussions, new curricula in mathematics for the secondary schools have been organized and written. The subject matter of these curricula can probably be more accurately described as "traditional mathematics which is developed from a more up-to-date point of view."

A more appropriate phrase to describe such a curriculum is *A Modern Program in Mathematics for the Schools*, since the curriculum is also being revised at the elementary level. In the same manner one could speak of *A Modern Course in Algebra, A Modern Course in Geometry*, etc. The Commission on Mathematics in their final report used the phrase *A Contemporary Program in Mathematics* to designate a program suggested to fit the needs of students in the second half of the twentieth century. They also used the term *Contemporary Mathematics* which has many of the drawbacks of the term *Modern Mathematics*. In this paper the terms "modern programs," "modern courses," "contemporary programs," and "contemporary courses" will be used interchangeably to indicate curricula and courses suggested for schools and students now in the schools.

[1] Vincent H. Haag, *Studies in Mathematics*, Vol. III, *Structure of Elementary Algebra* (School Mathematics Study Group, preliminary edition, 1960), pp. 1–5.

The criteria for a modern program in mathematics are, as suggested by Professor Phillip S. Jones, President of the National Council of Teachers of Mathematics, the answers to the following questions:

1 What topics will best help students to understand the nature, role and fascination of mathematics?
2 What topics will form the best basis for further studies of students, whether they be motivated by necessity or by curiosity?
3 What topics are most necessary to the future citizen's ability to understand his culture and to function effectively in it?
4 Which of all these topics can be most effectively studied at the student's level of maturity and in the time available?

The answers to these questions are by no means simple or easy. The answer to questions 1 and 2 must be based on a consideration of what mathematics is today, not merely on what is *new* in the mathematics of today.

The answer to question 2 must also be partially based on what the college expects of entering students. Presently the trend seems to be "toward expecting freshmen students to be prepared to begin with analytic geometry and the calculus."[2]

Proposed new programs in mathematics

The need for modernizing programs in mathematics and in mathematics teaching in the schools is not peculiar to Oklahoma. It is well recognized in many parts of the United States and something is being done about it in a great many schools. Though perhaps more work has been done in terms of courses and students involved at the secondary level, the movement is well defined and is moving forward at all levels from Grades 1 through college. Programs involving experimentation and the writing of teaching material are being initiated and carried on all over the country. One list of *Studies in Mathematics Education* includes descriptions of at least 30 such programs, including the program of the "Oklahoma State Committee on the Improvement of Mathematics Instruction."[3]

Because of this rather complete summary it will not be necessary here to discuss these programs in detail. From the standpoint of method and spirit of teaching they seem to indicate that we must:

1 Teach students so they will understand the principles of the mathematics with which they are working. Learning rules and methods of solving problems without understanding is not enough.
2 Emphasize the structure of mathematical systems as well as the deductive process by which the structure is built. All areas of mathematics may be used to do this, but variations must be made according to the maturity and previous preparation of the students involved.
3 Show the students that there are processes of discovery and creation followed by mathematicians before final reorganization into a more rigorously logical system. This again will vary with the maturity of the students, but should add to their understanding and appreciation of the subject as well as to their proficiency in skills and problem solving.[4]

As to content and organization of the school curriculum in mathematics, there are still wide differences of opinion. The aim of presenting and teaching substantial mathematics honestly, enthusiastically in a logically, understandably organized manner is probably shared by the various individuals and groups working in this field. It is recognized that every mathematics program must necessarily represent a deliberate choice of what is to be taught. It seems probable that different choices, if well organized, written and taught, may accomplish equally well the purpose of giving the students an understanding of the nature, role, and fascination of mathematics. It should be pointed out that many of the individuals and groups working on these programs take pains to indicate that they do not consider the teaching material which they are pre-

[2] Phillip S. Jones, "The Mathematics Teacher's Dilemma," *The University of Michigan School on Education Bulletin*, XXX (January, 1959), 65–72.

[3] *Studies in Mathematics Education. A Brief Survey of Improvement Programs for School Mathematics* (Chicago: Scott Foresman and Co., 1960.)

[4] Jones, *op. cit.*, pages 70–71.

senting as the only way to teach good mathematics to children. They are concerned with presenting *a way* which they hope will be successful. Common elements in the various recommendations seem to be:

1 Some breakup or scattering of topics throughout the curriculum. Almost certainly solid geometry will be included in the geometry course. The geometry course will include some reference to co-ordinate geometry; elementary trigonometry may be included in algebra and geometry; etc.
2 New approaches to the development of certain topics. Examples are the concepts of variables, relations and functions via the use of sets of ordered pairs of elements; a fuller and more adequate treatment of inequalities, absolute values, trigonometric functions as functions of numbers rather than angles; and graphical representations of all of these.
3 Certain new concepts with their terminology and symbolism introduced in such a way as to be genuinely useful in clarifying thinking in and understanding of elementary mathematics. Examples are set theory, such related ideas of modern logic as sentences, variables, and truth sets related to them.
4 The introduction of some topics, especially at the twelfth-grade level, which ordinarily appear in college courses. Such topics include a careful sequential treatment of probability and inferential statistics, some topics in modern algebra, matrix algebra, and the calculus. In a particular school some of these would be viewed as good enrichment and project material for superior students. It is generally agreed that more flexibility can be allowed at the twelfth-grade level but there is considerable disagreement as to just what the program should be.
5 Time for these changes in content and emphasis, which may be obtained by teaching two- and three-dimensional geometry as a unit, reducing the amount of time spent on the solution of oblique triangles in trigonometry, less emphasis on formal drill, including at least an introduction to algebra in the eighth grade, etc.

Experimental programs in mathematics in Oklahoma

There has been some individual experimentation going on in Oklahoma schools at all levels from Grades 1–12. This has been isolated in character as individual teachers and administrators tried to get something underway. It did not seem feasible, or particularly necessary, to try to collect full information on this. It does indicate that there has been a certain awareness of the need for a more adequate program in mathematics over the state.

However, the bulk of the experimentation with new programs in mathematics has resulted from co-operative efforts between the Oklahoma State Committee for the Improvement of Mathematics Instruction and the School Mathematics Study Group. During 1959–60 SMSG assigned 7 of its 49 Centers for Teaching SMSG Textbooks to this state. These involved the use in regular classes of textbooks for Grades 9, 10 and 11. Twenty-three school systems were involved with 42 teachers, 84 classes, and approximately 2,500 students. Eight college staff members served as consultants.

This program was considered very successful. Teachers reported much more interest and understanding on the part of these students than did those in classes taught from the traditional program. School administrators and parents were enthusiastic about the program and the results obtained. Students seemed to be as proficient in the standard skills of mathematics as those taking traditional courses, though no particular effort was made to make comparisons. Preliminary comparisons made by SMSG and the Minnesota National Mathematics Laboratory also indicate this is the case. Teachers reported that student ability in handling word or "story" problems was outstanding.

Examination of actual orders sent to SMSG for revised copies of SMSG textbooks for use in Oklahoma schools during 1960–61 reveals that 52 school systems will use the books in some or all Grades 7–12. The total number of texts ordered by Oklahoma schools for next year is 13,494. This exceeds five times the number of books used last year and again (as in the case of the Experimental Centers for 1959–60) Oklahoma's use of the books exceeds that of any state except California. The total number of orders in the entire nation was 110,921. Oklahoma's share was more than 12 per cent!

When it became known in the spring of 1960 that SMSG would write teaching material for mathematics in Grades 4, 5 and 6, Oklahoma was granted permission, through the influence of the State Mathematics Committee, to organize classes in 8 Points (of the total of 56 Centers and Points in the United States) to use the teaching material during 1960–61. These have been set up and are underway. They involve 64 teachers in some 25 elementary schools with 2,200 pupils.

Once it became apparent that a considerable amount of very carefully written teaching material would be produced over the country, the State Mathematics Committee decided that it could be most effective by making Oklahoma a proving ground for at least some of the material so produced. This decision was re-enforced by the realization that writing acceptable material in mathematics from a modern point of view takes time and skill as well as a thorough knowledge of mathematics itself. Busy teachers cannot be expected to do this sort of writing. They can render valuable assistance by trying out the material, offering constructive criticism, and doing rewriting if they can find time for it. In the future it may be possible to subsidize competent teachers and make available consultative service so that they can do some rewriting of this sort. The Committee expects to explore this possibility.

In Oklahoma, then, we have placed most emphasis on the teaching material, Grades 4–12, developed by the School Mathematics Study Group. There have been several reasons for this. We have been able to take advantage of SMSG's program of experimentation and the fact that some subsidy has been available. Other considerations were: (1) SMSG has complete teaching material for all grade levels; (2) a program initiated in a particular school can be continued into the next year; (3) a program can be expanded to include other schools in the state. The success of this idea is indicated by the large number of SMSG books ordered in Oklahoma this year. We are convinced that the books present good mathematics which children can understand and enjoy.

Summary and Implications

We may now draw some conclusions which may be helpful in interpreting and in judging the value (now and as it develops in the next few years) of the movement to modernize the program for mathematics in the schools. Improvements made will be the result of efforts made by many individuals—teachers, mathematicians, interested laymen, and organized groups of the entire mathematical community. Some organized groups have been able to make more rapid progress than others. This is particularly true of the School Mathematics Study Group which in two years' time has been able to organize, write, and test in the schools a complete set of textbooks for Grades 7–12 and to make a substantial beginning on a program for the elementary grades. This has been true because of superb leadership, their policy of taking full advantage of things done by other groups and individuals, and the fact that the group has been adequately supported by grants from the National Science Foundation. The following implications of this complete effort are listed:

1 It has been recognized that the need for revised contemporary courses in mathematics in the schools at all levels is urgent.
2 The programs proposed, while not presented as the *only* ways to solve this problem, do represent definite *ways* to make good mathematical content available to the schools.
3 The procedure adopted by most, if not all, of the groups and individuals is commendable, since it has been, from the start, a co-operative effort from all parts of the mathematical community. Trials in various kinds of classrooms with regular teachers, often over a period of years, have been the rule, rather than the exception.
4 The prospect of a complete curriculum with carefully written textbooks and teaching material is anticipated with enthusiasm by both teachers and administrators. However, the desire for an acceptable program should not lead to acceptance without a careful scrutiny of both content and method and point of view of teaching. All working with these pro-

grams recognize the need for constructive criticism on all phases of the program, and teachers have an obligation to exercise this right.

5 The experimental teaching and trials that have been done in connection with most of the proposed programs represents a significant testing of the teachability of the material under actual classroom conditions. An important element of this experimental work is that it involves, in a co-operative enterprise, the classroom teacher, the school administration, and mathematicians in college and universities.

6 The existence of teaching material written for classroom use under the conditions involved should make it possible for private and independent publishers to move rapidly into a complete revision or the new writing of commercial books. There are indications that this is already underway. One of the stated objectives of the SMSG is that their work will indicate a point of view useful in the writing of commercial textbooks. This may well be one of the most significant features of the whole program.

7 A modern program in mathematics established in the schools will tend to force the improvement of mathematics courses and the point of view in teaching them on the college levels. Such a revision is long overdue in the majority of colleges for students of mathematics, science, and engineering, as well as for prospective teachers of mathematics in both the elementary and the high school.

8 The availability of complete sets of textbooks and teaching materials for modern programs in mathematics for the schools, which are acceptable to the mathematical community, will go far to encourage action on modernization programs in this field at the state level.

This paper has dealt with objectives, methods of procedure, and the implications of recent efforts to develop complete contemporary programs in mathematics for the schools. These programs are not complete, but rapid progress is being made. They represent acceptable ways of teaching mathematics with understanding and interest, and constitute a long step in the solution of a very complicated problem. Those of us who are familiar with what appeared to be promising efforts over the last 50 years realize that success is often less than is anticipated. Perhaps the closer involvement of teachers at local levels may be the difference which will ensure the success of this program.

Letter to the editor

Dear Editor:

I submit the following comment on "Square Circles" by Francis Scheid in the May, 1961 issue of THE MATHEMATICS TEACHER:

Although this article is cleverly written, and although it has considerable merit in showing the pedagogic possibilities with interesting finite geometries, it is marred, in my opinion, by too great an emphasis on oxymoron.

On page 308 we find: "The locus of Figure 2 is unquestionably a *circle*, simply by definition." Which definition? If "circle" is defined as a set S, finite or infinite, of 'points', each of which is at a given 'distance' from a given point not in S', then, reluctantly or otherwise, we must accept the author's statement. The numerous possibilities in the primitives 'points' and 'distance' yield a variety of circle-concepts, one of which is the 12-point circle cited.

Towards the end of page 308 we read: ". . . where are all the points which are at a distance of three from A? . . . Figure 5 gives the picture. Behold the square circle!" We cite the justification for calling the figure a square: "These twenty-four points also form four straight line segments of length six, meeting at obvious vertices to form a quadrilateral. Since the diagonals of this quadrilateral are of equal length, *again six*, there is sufficient justification for calling it a square. Certainly it looks like a square."

Since, in the article, no explicit definition of 'square' is given, it is reasonable to assume that the author implies the usual high-school geometry definition of 'square', which, in one form or another, involves the concept of 'angle', a concept not present in the geometry being described.

Even more objectionable is the pictorial appeal, "Certainly it looks like a square." Pictorially, certainly Figure 5 does *not* look like a circle!

The mathematician's traditionally conservative and parsimonious use of words *intra* mathematics is commendable, but the indiscriminate application of Humpty-Dumpty's dicta about words must inevitably lead to difficulties and contradictions.

The unfortunate emphasis placed by the author on neologisms and antithesis, such as biangle, square circle, and square triangle, detracts, as I see it, from a very fine presentation of recursive relations and generating functions in the combinatorial problems posed.

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Abstracting, generalizing, and explaining–processes or relations?

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Perhaps psychologists have learned so little about "processes" like those considered herein because they are trying to study the elements of the null set.

When one reads books on educational psychology or methods of teaching, he finds defining, generalizing, abstracting, inferring, and explaining referred to as processes—mental activities carried on by a person. This conceptualization has existed for a long time, but during this time no great strides have been made in acquiring knowledge about these "processes," particularly the kind of knowledge of most value to a teacher, viz., the necessary and sufficient conditions for these "processes" to be carried on most expeditiously. Perhaps this is because the theoretical conceptualization is not a fruitful one. What I propose to do in this paper is to question the soundness of this conceptualization and then to offer a different conceptualization—one based on modern logic—which perhaps may be more fruitful in facilitating research than the one that has existed for more than fifty years.[1]

The logic of process words and achievement words

The conventional denotation of 'process' includes such activities as thinking, studying, running, digesting, eliminating, listening, and comparing. Ryle (5, 150)[2] calls the names of processes like these *task words.* Co-ordinate with task words in his theory are achievement words like 'win', 'choose', 'hear', 'find' (an object), 'solve', 'checkmate', and 'intuit'. There should be no necessary connotations of success with achievement words, for antitheses like 'lose', 'not choose', 'not hear', 'fail to solve' and other negations are also achievement words. More generally, 'succeed' and 'fail' are themselves achievement words.

Achievement words are names of episodes which, as Ryle says, can be dated but not timed. They signify more or less sudden climaxes or denouements. A universal generalization is proved when the last necessary statement in the proof has been given and not before. Think of the difference in significance between 'searching for', a task or process word, and 'finding', an achievement word. In applying an achievement word, we imply that over and beyond the performance there is a state of affairs distinct from the performance. For a runner to win he must not only run and

[1] The author wishes to acknowledge the help received in structuring the ideas of this paper from his discussions with Professors David Page and Herbert Vaughan.

[2] This notation indicates the entry in the reference list. The first number designates the book or essay and the second designates the page.

"hit the tape" ahead of anyone else, but he must also be declared the winner.[3]

Achievement words connote not just acts, exertions, performances or sequences of acts, exertions, or performances. They connote the fact that the acts, exertions, performances, or sequences of these have had certain results. Note that whereas we say a person has hunted for something assiduously or not assiduously, we do not say that he has found it assiduously or not assiduously. We describe a person as listening carefully, thoughtfully, or with disdain but not as hearing carefully, thoughtfully, or with disdain. More generally, adverbs properly used to modify task words are not used to modify achievement words.

To explicate further the point made in the previous paragraph, we expect a person who is thinking or arguing to tell us immediately and without considering evidence what he is doing. But we do not necessarily expect him to tell us immediately and without examining evidence that he has justified a decision, solved a problem, or proved a theorem. As Ryle says on one occasion (5, 151), "Achievements and failures are not occurrences of the right type to be objects of what is often, if misleadingly, called 'immediate awareness' " and on another occasion (5, 151), "To put it crudely, they [achievement words] belong not to the vocabulary of the player, but to the vocabulary of the referee."

One further difference in the logic of task words and achievement words is that the former properly have a present continuous tense but the latter do not. Among the more obvious instances of this are the achievement words 'know', 'believe', 'intuit' and 'discover'. The present continuous tense of these verbs simply makes no sense.[4] As another instance, although it makes sense to say, "Tom is running", it makes no semantic sense to say, "Tom is winning" even though Tom is twenty yards ahead of the nearest runner and the tape is only a yard ahead of him.[5] Similarly, whereas it makes semantic sense to say, "Tom is trying to prove Theorem 6", it makes no semantic sense to say, "Tom is proving Theorem 6".

"Processes" conceived of as relations

All this lays the foundation for what I propose to argue; viz., that terms like abstracting, generalizing, inferring, explaining, and justifying (in the sense of giving reasons to support a claim to knowledge, a belief, or an action) are not fruitfully to be regarded as process or task words, that is, as names of psychological processes. Their logic is more akin to that of achievement words. Consider inferring. An autobiographical account of what the person is thinking about or feeling does not by itself indicate that he is inferring until he comes to a conclusion—the identifiable characteristic of inferring. One does not generalize until a generalization is reached, abstract until the properties common to all the elements in the set considered are recognized, or infer until a conclusion is reached. It is not simply a matter of doing something. The doing must have an upshot, have had certain results, or culminated in a product that satisfies certain criteria. While "I am thinking", "You are teaching", and "He is studying" signify something which cannot be signified as well in another way (save by equivalent expressions in another language), "I am

[3] "Hitting the tape" first is certainly not the same as winning, for there is the possibility of being disqualified for some violation of a rule.

[4] A use such as "I am discovering that if one does not master the basic concepts, he will have trouble later on" is not a counterexample. It merely indicates an impoverished vocabulary or mental lethargy in hunting for a proper word.

[5] To be sure, people *do* say, "Tom is winning". But upon being queried as to what they mean, they probably would say, "I mean Tom is ahead and probably will be the winner". We do not always have to *say* what we mean for people to *know* what we mean. Mathematics provides lots of evidence to support this generalization. "Add $2a+3a$" and "invert the divisor" are nonsense both semantically and mathematically. But they are not nonsense pragmatically, for both teachers and students behave appropriately with respect to them. The point is that when we consider what is *said*—not what is *meant*—the illogical nature of the sentence becomes apparent.

generalizing", "You are explaining", and "He is abstracting" signify nothing more than "I generalize" (or generalized), "You explain" (or explained), and "He abstracts" (or abstracted).

In light of the foregoing, it seems reasonable to argue that the logic of terms like 'explaining', 'inferring', 'generalizing', 'abstracting', and 'justifying', which are used in talking about learning and teaching, is sufficiently different from that of terms used to denote cognitive processes as to warrant a different theoretical treatment. The theory of relations permits such a treatment. A relation is a set of ordered n-tuples; a binary relation is a set of ordered pairs of elements. By proper grouping of the components of the members of some relations, we can convert them into binary relations. The domain of a binary relation is the set of first components of the pairs and the counterdomain or range is the set of second components of the pairs. If A and B denote sets and R a relation between A and B (more precisely, a relation in $A \times B$), we write

$$R = \{(x, y) \mid x \in A \text{ and } (y = R(x)) \in B\}.$$

x is replaceable by (a name of) an element of the domain of R, viz., set A, and y is replaceable by (a name of) an element of the range of R, viz., set B.

Let us now see how the theory of binary relations can be used to explicate abstracting, generalizing, explaining, justifying, and inferring—activities that predominate in all instruction that is cognitive in nature.

y *is an abstraction from* x. When we abstract, we look for similarities amid differences. Given a set of things, we identify the properties that are common to all the members of the given set. Hence the domain of this relation is the set of nonsingular sets of entities. The range is the set of properties. For any value of x, e.g., $\{3, 4, 5, 6, 7, 8\}$, the value of y is the set of properties common to all elements of the set which replaces x, viz., being a natural number greater than 2 and less than 9.

y *is a generalization of* x. To take an example, 'All persons in Illinois are healthy' is a generalization of 'All persons in Chicago, Illinois, are healthy' which, in turn, is a generalization of 'Fifty persons in Chicago are healthy'. 'All persons in Illinois are healthy' is also a generalization of 'Most high school students in Illinois are healthy' and also of 'Some men in Illinois are healthy' provided by 'some' we mean at least one but not all.

It can be seen that the domain of this relation is a proper subset of the set of statements. If the first component (value of x) of an element of the relation is a statement, the second component (value of y) is also a statement and the generalizing relation is a subset of the Cartesian product statements$\times$statements.

Let us define *test set of a statement* as the set of entities used to determine the truth-value of the statement. For example, the test set of the statement, All high schools in the United States at present offer at least one year of mathematics, is the set of all high schools in the United States currently in existence. By using the concept of the test set of a statement, we can see a difference between the two components of a member of the generalizing relation. The test set of the second component is a superset of the test set of the first component.

We also generalize using exhortative sentences—sentences containing 'should' or 'ought'; e.g., 'Teachers should keep abreast of the changes in the subjects they teach' and 'We ought not discriminate against people because of their race or religion'. If the first component of an element of the generalizing relation is an exhortative sentence, the second component is also an exhortative sentence. As in the case of pairs of statements, the set to which the second component is applicable is a superset of the set to which the first component is applicable.

y *explains* x or x *is explained by* y where explains = clarifies. In the context of teaching, the term 'explain' is used ambig-

uously. There appear to be three distinct uses:

1 A descriptive use as when a teacher says, "I'll explain how to solve a linear equation" and then describes what to do to solve the equation. Or he gives a sequence of prescriptions; e.g., "First, get all the terms containing the unknown on one side of the equals sign; second, collect like terms; third, divide both sides by the coefficient of the unknown," and so on.
2 A definitional use as when a teacher says, "Tom, will you please explain what the term 'relation' means." Tom gives a correct definition of the term and the teacher is satisfied with the answer.
3 A subsumptive use as when a teacher says, "Will you explain why the set $\{(1, 0), (2, 1), (3, 2), (2, 0), (4, 1)\}$ is not a function."

The two latter uses are the interesting ones for the present discussion. To keep the ideas straight, I shall employ the technique of using different names for each of the relations. I shall call the most general relation: y clarifies x and then separate this into two subrelations: (1) y is a definition of x; and (2) y explains x. Explication of each of these will largely explicate the more extensive relation y clarifies x.

y *is a definition of* x or x *is defined by* y. The domain of this relation is the set of terms and the range is uses of the *definiendum*—the term being defined. The range can be partitioned into three subsets determined by the three uses of a term: (1) to denote things named by the term, (2) to connote properties possessed by the *denotata*, and (3) to imply other terms. Each of these can be regarded as a relation and the relation described using the concepts of domain and range identified above.

y *explains* x or x *is explained by* y. 'Explain' is used in the sense of telling why. As Hospers has pointed out (4), every case of this kind of explanation is a case of subsumption. The *explanandum*—what is explained—is shown to be a particular instance of a generalization. Sometimes the generalization is tacitly implied by the subsuming statement as in the case of:

Teacher: "Why are triangles ABC and DEF congruent?"
Student: "Because each side of $\triangle ABC$ is congruent to a corresponding side of $\triangle DEF$."

The generalization that is implied might be stated as, "If each side of a triangle is congruent to a corresponding side of a second triangle, then the triangles are congruent."

Sometimes the generalization is stated and the subsuming statement is tacitly implied as in the case of:

Student 1: "Why can we write an equals sign between '6+4' and '10'?"
Student 2: "If two numerals denote the same number, we can write an equals sign between them."

The subsuming statement might be stated as " '6+4' and '10' denote the same number."

Each element of this relation is an ordered pair whose first component (value of x) is either a phenomenon or a mathematical "fact" and whose second component (value of y) is a set of reasons. The second component may be a singular set as when a person gives only one reason why something is the case, or it may be a nonsingular set as when a person gives more than one reason, often both the generalization and the subsuming statement. If both the generalization and subsuming statement are given as reasons, the value of x necessarily follows from the corresponding value of y. An example is: (Jones died., {If a person eats a large amount of bichloride of mercury, he dies., Jones ate a large amount of bichloride of mercury.}) $\in$ explanations.

By a restriction of the field of the relation, one can distinguish between true explanations, that is, those in which the reasons both stated and implied are true; false explanations, that is, those in which at least one of the reasons is false; and explanations consisting of reasons, the truth-value of one of which is unknown.

y *justifies* x or x *is justified by* y. Justification consists in giving reasons to support a knowledge claim, a belief, an action either proposed or accomplished, or a decision. Following Feigl (1), the set of justifications can be partitioned into two subsets: (1) justifications of claims to knowledge or beliefs concerning the truth-value of a statement, or (2) justifications of beliefs concerning what is right, good, beautiful, or ought to be done, and justifications of decisions that have been made or actions that have been performed. Feigl calls the former *validations*, the latter *vindications*. If each of these relations is explicated, the union of the two (namely, y justifies x) is thereby explicated.

y *validates* x or x *is validated by* y. Examples of this eventuate when one answers the questions, "How do you know that Illinois is east of Iowa?" or "How do you know that $\sqrt{2}$ is not a rational number?" Each of the elements of this relation is an ordered pair whose first component is a statement such that the second component is a set of statements serving as evidence or grounds for the first component. If the first component is a contingent statement, the second component will be a set of contingent statements. If the first component is an analytic statement (more precisely, a statement which we expect to prove to be analytic), the second component will be a set of analytic statements—theorems or postulates—or definitions. In this case validation is identical with proof.

y *vindicates* x or x *is vindicated by* y. In this relation x is replaceable by a sentence in which something is rated by means of words like 'right', 'good', or 'beautiful' or their opposites; or the words 'ought' or 'should' are used to indicate a preference for a particular action. Examples of possible values of x are 'Mathematics ought to be taught in the schools', 'Lying is wrong', 'The Mona Lisa is a beautiful painting', and 'It is a good idea to check your answers'.

y is replaceable by sets of statements serving as evidence or grounds for x. As in the case of validation, a set which is a value of y may be a singular set—a set consisting of only one statement; or the set may be nonsingular as is often the case when involved vindications are attempted.

y *is inferred from* x. Replacements of y (that is, the range of the inferring relation) come from the set of statements or quasi statements. The latter are sentences that are sometimes used to make a statement (e.g., 'This proof is elegant', where 'elegant' has a precise definition) and hence have a truth-value. But they are not always used to make a statement, that is, to describe some state of affairs. Sometimes they are used to make a value judgment; e.g., Equality of educational opportunity is right. Or they are used to suggest what ought to be done; e.g., One year of mathematics should be required for graduation from high school.

Replacements of x (that is, the domain of the inferring relation) come from the set of sets of statements offered as reasons for y. Values of x may be singular sets or nonsingular sets.

Note that the inferring relation is described so as to include inferences which cannot be justified by any valid inference formula as well as inferences which can be justified by showing that they fit some valid inference formula. This was done because people use both kinds.

It should also be noted that generalizing, explaining, and justifying are subrelations of inferring.

Justification of this Conceptualization

To someone who appreciates structure, the preceding theory may be appealing. It

places concepts in a subsumptive and deductive organization. Moreover, it permits employment of the theory of relations. For example, the converses of abstracting, generalizing, and inferring can be interpreted respectively as exemplifying, restricting, and implying (as this term is popularly used). To take another example, since we know how to determine whether or not two relations are identical, we know that the relation of meaning is not the same as the relation of picturing. Yet some of the advice proffered on teaching reading seems to be based on the assumption that these two relations are identical. And some of the explications of meaning one hears in discussions and reads in articles also seem to be based on this same gratuitous assumption.

The properties of reflexivity, symmetry, and transitivity can be used to clarify various relations. For example, those relations whose domains and ranges are disjoint (e.g., abstracting, denoting, and connoting) are irreflexive and asymmetric. Generalizing in the set of statements is irreflexive, asymmetric, and transitive. The relation x means the same as y in the set of terms is reflexive, symmetric, and transitive and hence is an equivalence relation. The relation x means y is irreflexive and asymmetric and hence not an equivalence relation.

The theory is not inimical to learning theory. It is particularly appropriate to a sophisticated theory like Guilford's (2), (3). Guilford conceives the intellect essentially as the triple Cartesian product: products$\times$contents$\times$operations. Space does not permit an elaboration of his theory. Suffice it to say that to him products denote units, classes, relations, systems, transformations, and implications; contents are figural, symbolic, semantic, or behavioral; and operations consist of cognition, memory, divergent thinking, and convergent thinking. While he uses the layman's concept of a relation, the conceptualization I have proposed could easily be incorporated in his theory and should enhance it.

Existing pedagogical theory is diffuse, lacks structure, and is beset with semantic confusion. It may just be that these are among the reasons for such slow progress in research in methods of teaching. Perhaps psychologists have learned so little about "processes" like those considered herein because they are trying to study the elements of the null set.

Will this conceptualization result in more definite research? I do not know at present. But I do know that those areas of empirical knowledge which are characterized by the greatest progress in research are also characterized by clearly formulated and structured theories. There is scarcely anything as fruitful as a good theory, for it is the source of ideas—hypotheses that lend themselves to testing empirically. Perhaps a different conceptualization of abstracting, generalizing, explaining, *et al.* may be the catalyst we need for definitive research.

References

1. Feigl, Herbert, "Validation and Vindication: An Analysis of the Nature and Limits of Ethical Arguments," *Readings in Ethical Theory*, edited by Wilfred Sellars and John Hospers. New York: Appleton-Century-Crofts, Inc., 1952, pp. 667–80.
2. Guilford, J. P., "Three Faces of Intellect," *The American Psychologist*, XIV (1959), 469–79.
3. Guilford, J. P., "The Structure of the Intellect," *Psychological Bulletin*, LIII (1956), 267–93.
4. Hospers, John, "What is Explanation?," *Essays in Conceptual Analysis*, edited by Antony Flew. London: Macmillan Co., Ltd., 1956, pp. 94–119.
5. Ryle, Gilbert, *The Concept of Mind*. London: Hutchinson's University Library, 1955.

Probability and the radioactive disintegration process

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An application using the mathematical topics of probability, the binomial theorem, and the number e

IT HAS BEEN USEFUL in the teaching of science or mathematics to consider the radioactive disintegration process from the standpoint of simple probability theory and to arrive at the mathematical expression for this process.

In the radioactive disintegration process specific elements undergo a spontaneous breakdown. Different elements from the "parent" radioelement appear and are usually, themselves, radioactive, and they in turn disintegrate to produce still other elements. The process is due to the instability of the nuclei involved. The rate of disintegration or decay varies with the particular radioactive element. Some radioactive elements have a slow rate of decay (millions of years) and others survive for only a fraction of a second.

It has been pointed out (1, 2, 3, 4) that the radioactive decay process can be described in terms of probability. If we let p equal the probability that a certain atom of a radioelement will decay in a given time interval, Δt, and assume that p is dependent *only* on the length of the time interval, Δt, then

$$p \text{ is proportional to } \Delta t \tag{1}$$

if Δt is a short interval of time. We can then say that

$$p = k\Delta t \tag{2}$$

where k is a proportionality constant and depends on the radioactive atom in question.

If the disintegration process is to be considered from the standpoint of probability, we know from probability theory that the probability of a radioactive element *not* disintegrating in the Δt interval is $(1-p)$. The probability of the atom not disintegrating in the *next* Δt period is also $(1-p)$. We also know that the probability for the given atom to survive the first and second intervals of time, Δt, is given by

$$(1-p)^2. \tag{3}$$

If $p = k\Delta t$, then (3) becomes

$$(1-k\Delta t)^2. \tag{4}$$

Now if we consider n such intervals of time, the probability for survival becomes

$$(1-k\Delta t)^n. \tag{5}$$

If we consider n intervals each of length Δt then the total time, T, is equal to $n\Delta t$. Using this equality, we can substitute in (5) and obtain

$$\left(1-\frac{kT}{n}\right)^n \tag{6}$$

as the probability that the atom will remain unchanged after time T.

We can readily see from (5) and/or (6) that when $\Delta t \to 0$ we can say that the probability that the atom will remain unchanged in time T is the

$$\lim_{n\to\infty}\left(1-\frac{kT}{n}\right)^n. \tag{7}$$

This leads to an accurate value for the probability since n becomes very large.

Now we know that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$
$$\equiv e = 2.71828 \cdots \text{ (see Fig. 1)}, \quad (8)$$

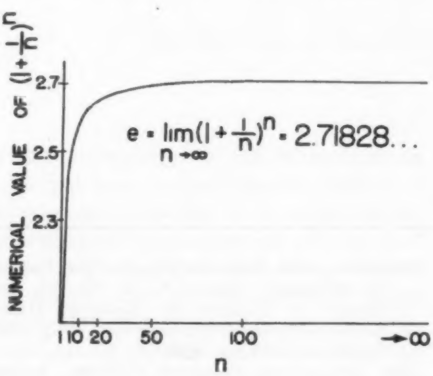

Figure 1

and therefore that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x, \quad (9)$$

or that

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}. \quad (10)$$

Therefore, from (7) and (10) we can write that

$$\lim_{n\to\infty}\left(1-\frac{kT}{n}\right)^n = e^{-kT}. \quad (11)$$

If we are dealing with many radioactive atoms, say N, the fraction remaining *unchanged* after time T will be N_t/N if N_t is the number of unchanged atoms in time T. This means, therefore, that

$$N_t/N = e^{-kT}$$

or

$$N_t = Ne^{-kT}. \quad (12)$$

If we plot the number of atoms surviving against the elapsed time we get the typical exponential curve shown in Figure 2. From this figure we can tell the time required for any particular fraction of the total number of atoms to decay. A convenient fraction is $N_t/N = \frac{1}{2}$, that is, the time required for the decay of one-half of the atoms of a sample. This time is the familiar term "half-life" of a particular radioactive element.

If we take the natural logarithm of both sides of equation (12) we have

$$\ln\left(\frac{N_t}{N}\right) = -kT. \quad (13)$$

Now if $N_t/N = \frac{1}{2}$, then

$$\ln \tfrac{1}{2} = -kT$$

or

$$\ln 2 = kT. \quad (14)$$

Since $\ln 2 = 0.693$, we can write (14) as

$$T = \frac{0.693}{k}\cdot \quad (15)$$

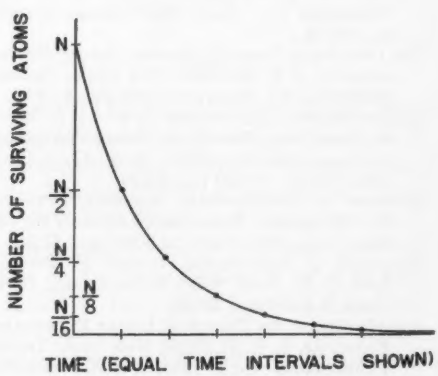

Figure 2

In summary, simple probability theory has been used to arrive at the mathematical expression for the radioactive disintegration process. In this approach, e has been introduced. A third, simple but important topic, the binomial theorem, can be utilized to verify (9) and (10). In addition, the relationship between the half-

life and the constant k (equation 15) is arrived at in a simple manner.

REFERENCES

1. Evans, Robley D., *The Atomic Nucleus.* New York: McGraw-Hill Book Co., Inc., 1955, pp. 470 ff.
2. Friedlander, Gerhardt and Kennedy, Joseph W., *Nuclear and Radiochemistry.* New York: John Wiley & Sons, Inc., 1955, pp. 6 ff.
3. Hamill, William H. and Williams, Russell R., Jr., *Principles of Physical Chemistry.* Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1959, pp. 430 ff.
4. Overman, Ralph T. and Clark, Herbert M., *Radioisotope Techniques.* New York: McGraw-Hill Book Co., Inc., 1960, pp. 285 ff.

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BOOKS

COLLEGE

Advanced Calculus, An Introduction to Analysis, Watson Fulks. New York: John Wiley & Sons, Inc., 1961. Cloth, xv+521 pp., $11.25.

Analytic Geometry and Calculus, Herbert Federer and Bjarni Jonsson. New York: The Ronald Press Co., 1961. Cloth, vi+671 pp., $8.75.

Basic Concepts in Modern Mathematics, John E. Hafstrom. Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1961. Cloth, x+195 pp., $5.75.

The Calculus of Finite Differences, George Boole, edited by J. F. Moulton. New York: Chelsea Publishing Co. Paper, xii+336 pp., $1.39.

College Business Mathematics (3rd ed.), R. Robert Rosenberg. New York: Gregg Publishing Division, McGraw-Hill Book Co., Inc., 1961. Cloth, x+420 pp., $5.50.

Elements of Mathematical Statistics, Howard W. Alexander. New York: John Wiley & Sons, Inc., 1961. Cloth, xi+367 pp., $7.95.

Essentials of Mathematics, Russell V. Person. New York: John Wiley & Sons, Inc., 1961. Cloth, x+646 pp., $7.00.

Introduction to the Theory of Linear Differential Equations, E. G. C. Poole. New York: Dover Publications, Inc., 1960. Paper, viii+202 pp., $1.65.

Measure, Lebesgue Integrals, and Hilbert Space, A. M. Kolmogorov and S. V. Fomin. Translated by Natascha Artin Brunswik and Allan Jeffrey. New York: Academic Press, 1961. Cloth, xii+147 pp., $4.00.

Partial Differential Equations and Continuum Mechanics, edited by Rudolph E. Langer. Madison, Wis.: The University of Wisconsin Press, 1961. Cloth, xv+397 pp., $5.00.

Probability: A First Course, Frederick Mosteller, Robert E. K. Rourke, and George B. Thomas, Jr. Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1961. Cloth, xv+319 pp., $5.00.

Probability with Statistical Application, Frederick Mosteller, Robert E. K. Rourke, and George B. Thomas, Jr. Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1961. Cloth, xv+478 pp., $6.50.

Sets, Logic, and Axiomatic Theories, Robert Stoll. San Francisco: W. H. Freeman & Co., 1951. Paper, x+206 pp., $2.25.

Theory of Maxima and Minima, Harris Hancock, New York: Dover Publications, Inc., 1960. Paper, xiv+189 pp., $1.50.

A Treatise on the Calculus of Finite Differences (2nd ed.) George Boole, edited by J. F. Moulton. New York: Dover Publications, Inc., 1960. Paper, xii+336 pp., $1.85.

HIGH SCHOOL

Algebra One, Dustan Hayden and E. J. Finan. Boston: Allyn and Bacon, Inc., 1961. Cloth, viii+440 pp., $4.32.

High School Mathematics: Unit 7—Mathematical Induction, University of Illinois Committee on School Mathematics. Urbana, Illinois: University of Illinois Press, 1961. Paper, student's edition, 156 pp., $1.25; teacher's edition, $2.75.

Modern Algebra, First Course, Richard E. Johnson, Lona Lee Lendsey, and William E. Slesnick. Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1961. Cloth, ix+628 pp., $5.28.

Modern Elementary Algebra, Eugene D. Nichols, and Wagner G. Collins. New York: Holt, Rinehart, & Winston, Inc., 1961. Cloth, ix+402 pp., $4.08.

Using Mathematics 7, modern topics edition, Kenneth B. Henderson and Robert E. Pingry. New York: McGraw-Hill Book Co., Inc., 1961. Cloth, xii+468 pp., $4.20.

Using Mathematics 8, modern topics edition, Kenneth B. Henderson and Robert E. Pingry. New York: McGraw-Hill Book Co., Inc., 1961. Cloth, $4.20.

Trammel method construction of the ellipse

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An extension of the trammel method for construction of an ellipse

THE CONSTRUCTION of the ellipse from its principal axes by a trammel method is widely used. One way of carrying out the construction is by laying off on a strip of paper half the minor axis, AB, and half the major axis, BC; letting C slide along the minor axis and A along the major axis, and marking the positions of B to form the ellipse. However, the extension of this method to the case where conjugate diameters are given is not so well known. In fact, as far as we are aware, this methd is not encountered in any books in English on analytic geometry or on engineering drawing. In a note by Carl Rodenburg in "Zeitschrift für Mathematik und Physik," Bd. XXIX (1884), p. 255, a trammel method construction is given, however without analytic proof. We give here a somewhat different procedure and include an analytic proof of the construction.

CONSTRUCTION

Let there be given two conjugate diameters A_1A and B_1B of a required ellipse, making an angle ϕ and intersecting at O, as shown in Figure 1, and with $B_1B < A_1A$. From B draw a perpendicular to A_1A intersecting it at D. On DB extended lay off $BC = OA$. Connect C with O and extend it as needed (say CC_1). On the edge of a strip of paper mark D', B', C', coinciding with D, B, C, respectively. If C' slides on CC_1 and D' slides on AA_1 then B' describes the required ellipse.

The principal axes of the ellipse are determined as follows: On A_1A lay off

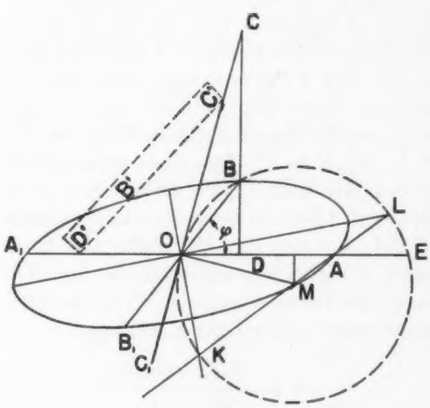

Figure 1

$OE = DC$. With OE as chord draw a circle tangent to CC_1 at O whose center, M, is found as the intersection of the perpendicular bisector of OE and the perpendicular to CC_1 at O. Connect M and A by a line intersecting the circle at K and L, where L is on the same side of M as A so that $AL < AK$. Then (*a*) AL and AK are half the minor and major axes of the ellipse, respectively, and (*b*) OK and OL are their respective directions.

If the two given conjugate diameters should coincide with the principal axes of the ellipse, this construction would then become the well-known trammel construction of the ellipse from principal axes, mentioned above. In this case while the circle construction for the principal axes could still be carried through, it would be unnecessary since the lengths of the principal axes are known to begin with. Con-

sequently this case is not considered in the following; i.e., it is assumed $\phi \neq \pi/2$ and, in fact, for convenience we take $\phi < \pi/2$.

We now turn to the proof of the correctness of the constructions. First we show that the locus is an ellipse; next the lengths and directions of the principal axes are found; and then the circle construction is considered.

Equation of the locus

Given $A_1A = 2a$ and $B_1B = 2b$ with O the mid-point of each line. As mentioned above $\angle AOB = \phi < \pi/2$. In Figure 2 let the designations of Figure 1 still hold and let the line NPR represent some general position of the strip $D'B'C'$, with P, the point on the locus, corresponding to B', N to D', R to C'. Introduce rectangular co-ordinate axes (x, y) with origin at O and with the plus x-axis in the direction of OA. Let the line OC make the fixed angle ψ with the x-axis and the line NR make the variable angle β. Write the constant length $NR = DC = \lambda$ and thus

$$\cot \psi = \frac{OD}{DC} = \frac{b \cos \phi}{\lambda}.$$

Next introduce $m = b \sin \phi = DB = NP$ then $\lambda = DB + BC = m + a$. Let T and S be the projections of R and P respectively on the x-axis. We then have $NT = \lambda \cos \beta$

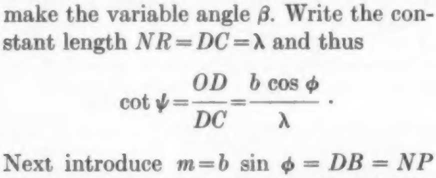

Figure 2

and $TR = \lambda \sin \beta$ so that $OT = TR \cot \psi = \lambda \sin \beta \cot \psi = b \sin \beta \cos \phi$ and

$$\begin{aligned} NO &= NT - OT \\ &= \lambda \cos \beta - b \sin \beta \cos \phi. \end{aligned}$$

Thus we get

$$\begin{aligned} x &= OS = ON + NS \\ &= (b \sin \beta \cos \phi - \lambda \cos \beta) + m \cos \beta \end{aligned}$$

or, since $\cos \beta = \sqrt{m^2 - y^2}/m$ and $\sin \beta = y/m$,

$$x - \frac{by \cos \phi}{m} = (m - \lambda)\sqrt{m^2 - y^2}/m. \quad (1)$$

Squaring (1), taking into account that $\lambda - m = a$, we obtain after some manipulations

$$\frac{x^2}{a^2} - 2xy\frac{\cot \phi}{a^2} + y^2\left(\frac{\cot^2 \phi}{a^2} + \frac{\csc^2 \phi}{b^2}\right) - 1 = 0, \quad (2)$$

corresponding to

$$Lx^2 + 2Mxy + Ny^2 + H = 0.$$

This is the equation of the locus, an equation of the second degree. The discriminant of this expression is

$$\begin{aligned} M^2 - LN &= \frac{\cot^2 \phi}{a^4} - \frac{\cot^2 \phi}{a^4} - \frac{\csc^2 \phi}{a^2 b^2} \\ &= \frac{-\csc^2 \phi}{a^2 b^2} < 0. \end{aligned} \quad (3)$$

Hence the curve is an ellipse, and since A and B are actual points of the curve (as readily seen by substitution of the co-ordinates of $A(a, O)$ and $B(b \cos \phi, b \sin \phi)$ into (2)), it is a real ellipse.

Lengths of the semi-axes of the ellipse

On rotation of axes to remove the xy term, the equation (2) of the ellipse with respect to the new co-ordinate axes x', y', becomes

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1,$$

corresponding to

$$L'x'^2+2M'x'y'+N'y'^2+H=0.$$

The discriminant of this expression is

$$M'^2-L'N'=-\frac{1}{a'^2b'^2},$$

while

$$L'+N'=\frac{a'^2+b'^2}{a'^2b'^2}.$$

Equating this to the discriminant (3) we have

$$1/a'^2b'^2=(\csc^2\phi)/a^2b^2$$

or

$$a'^2b'^2=a^2b^2\sin^2\phi. \tag{4}$$

From (2) we have

$$L+N=\frac{1}{a^2}+\frac{\cot^2\phi}{a^2}+\frac{\csc^2\phi}{b^2}=\frac{a^2+b^2}{a^2b^2}\csc^2\phi.$$

On equating this to $L'+N'$, we get

$$\frac{a^2+b^2}{a^2b^2}\csc^2\phi=\frac{a'^2+b'^2}{a'^2b'^2}$$

or by virtue of (4)

$$a^2+b^2=a'^2+b'^2. \tag{5}$$

From (4) and (5) we see that a'^2 and b'^2 are the roots of the quadratic equation

$$z^2-(a^2+b^2)z+a^2b^2\sin^2\phi=0.$$

Therefore, if $a'>b'$, we have

$$a'^2=(a^2+b^2+\sqrt{a^4+2a^2b^2\cos 2\phi+b^4})/2, \tag{6}$$

$$b'^2=(a^2+b^2-\sqrt{a^4+2a^2b^2\cos 2\phi+b^4})/2. \tag{7}$$

Directions of the axes

The angle of rotation of axes which removes the xy term from (2) is the angle between a principal axis and an original coordinate axis. By the usual formula, this angle is found from

$$\begin{aligned}\tan 2\alpha&=\frac{-2\cot\phi}{a^2}\Big/\left[\frac{1-\cot^2\phi}{a^2}-\frac{\csc^2\phi}{b^2}\right]\\&=-2b^2\sin\phi\cos\phi/(b^2\sin^2\phi\\&\quad -b^2\cos^2\phi-a^2)\\&=b^2\sin 2\phi/(a^2+b^2\cos 2\phi).\end{aligned} \tag{8}$$

Circle construction

According to the construction given below, the points A and E have the co-ordinates $A(a, O)$, $E(\lambda, O)$. The equation of the yerpendicular bisector of OE is $x=\lambda/2$, while the equation of the perpendicular to CO at O is

$$y=-x\cot\psi=(-xb\cos\phi)/\lambda.$$

Hence the center M of the circle tangent to CO at O and passing through E, i.e., the intersection of the above two lines, has the co-ordinates $x=\lambda/2$, $y=(-b\cos\phi)/2$.

The diameter KL is equal to $2OM$, thus

$$\begin{aligned}KL&=2\sqrt{\frac{\lambda^2}{4}+\frac{b^2}{4}\cos^2\phi}\\&=\sqrt{a^2+2ab\sin\phi+b^2}\end{aligned}$$

since

$$\lambda=a+b\sin\phi.$$

Now,

$$AL\cdot AK=AO\cdot AE=ab\sin\phi$$

and

$$AL+AK=KL=\sqrt{a^2+2ab\sin\phi+b^2},$$

so that AL and AK are the roots (the smaller and larger respectively) of the quadratic equation

$$z^2-\sqrt{a^2+2ab\sin\phi+b^2}\,z+ab\sin\phi=0.$$

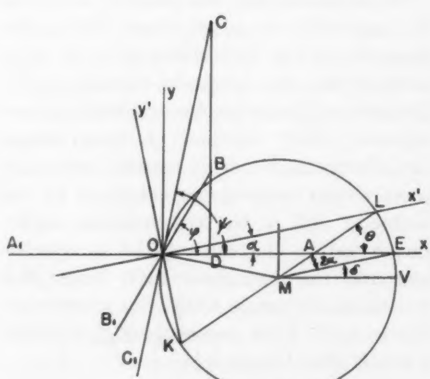

Figure 3

If z_1 and z_2 are the two roots, then

$$z_1, z_2 = (\sqrt{a^2+2ab\sin\phi+b^2} \pm \sqrt{a^2-2ab\sin\phi+b^2})/2$$

and their squares

$$z_1^2, z_2^2 = (a^2+b^2 \pm \sqrt{a^4+2a^2b^2\cos 2\phi+b^4})/2.$$

Comparing this expression with (6) and (7), we see that the assertion (a) is proved.

Now from the figure we see $2\alpha=\theta-\delta$ where $\theta=\angle(x, MA)$ and $\delta=\angle(x, ME)$. Furthermore we have $\tan\delta=-$slope of OM, so

$$\begin{aligned}\tan\delta &= (\tfrac{1}{2}b\cos\phi)/\tfrac{1}{2}\lambda\\ &= (b\cos\phi)/(a+b\sin\phi)\end{aligned}$$

and $\tan\theta=$slope of AM, so

$$\begin{aligned}\tan\theta &= (\tfrac{1}{2}b\cos\phi)/(a-\tfrac{1}{2}\lambda)\\ &= (b\cos\phi)/(a-b\sin\phi).\end{aligned}$$

Hence,

$$\begin{aligned}\tan 2\alpha &= \tan(\theta-\delta)\\ &= (b^2\sin 2\phi)/(a^2+b^2\cos 2\phi).\end{aligned}$$

On comparison with (8), we see that OL is the direction of one of the principal axes of the ellipse, and hence OK is the direction of the other principal axis.

Major and minor axes

We now complete the proof of assertion (b); namely, we prove that OL is the direction of the major axis. For all positions of $NR=\lambda$, the circles through O, N, R are equal, being circles in which a given segment (NR) subtends a given angle (ψ). Furthermore, it is readily seen that these circles have diameters equal to the diameter KL of the construction circle, i.e., equal to the sum of the semimajor axis and the semiminor axis. When NR moves in the region COE, the greater arc of the circle NOR passes through O, since $\psi<\pi/2$. (See Figure 4.)

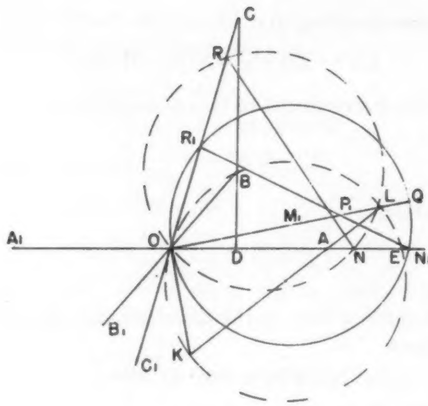

Figure 4

The segment OP always represents the radial distance of the point P of the ellipse from the center. Now when P lies on the line OL, a principal axis of the ellipse, at P_1, the length of OP_1 must be either a maximum or a minimum. Hence OL will have the same direction as the diameter OQ of the circle O, N_1, R_1, (center M_1). Since R_1ON_1 is the greater arc, the segment OP_1 is greater than the segment P_1Q. Hence OP_1 must be the maximum radial distance (and so P_1Q is the minimum) and thus OL gives the direction of the major axis. Consequently, OK gives the direction of the minor axis.

Conjugate diameters

Finally, it can readily be seen that the fundamental lines, AA_1 and BB_1, of this construction are conjugate diameters of the ellipse

$$x'^2/a'^2+y'^2/b'^2=1.$$

To recognize this use the theorem: The sum of the squares of any two conjugate diameters of an ellipse is constant and equal to the sum of the squares of the axes.[1]

[1] Cf., Briot et Bouquet, *Leçons de Géometrie analytique* (Paris, 1880), p. 156; or D. M. Y. Sommerville, *Analytical Conics* (London, 1924), p. 43.

Edited by Eugene D. Nichols, Florida State University, Tallahassee, Florida

UICSM's decade of experimentation*

William T. Hale, Assistant Director, UICSM Project, University of Illinois, Urbana, Illinois

Imagine that you are visiting a beginning algebra class in one of the schools using the materials prepared by the University of Illinois Committee on School Mathematics (UICSM). Here is a discussion which you might overhear.

Teacher: We have discovered that the numbers with which you have been working in earlier grades are not adequate for dealing with certain problems—problems where both an amount of change and a direction of change are involved. To discover that a number system is inadequate is not a new experience, I'm sure.
When you first started school you could probably count and perhaps do a little addition or subtraction using what we now call 'whole numbers'. But when you met a problem, such as:

If two boys work together on a job and are paid five dollars, how much should each get?

you discovered that your "whole numbers" were inadequate. You then learned about a new type of number called 'a fractional number'. What did you do with these numbers after you learned about them?

Joe: We learned to add, subtract, multiply, and divide them.

Teacher: That's right. You learned what people meant when they spoke about the sum, difference, product, or quotient of two fractional numbers.

Mary: And we used pies and sticks to help us.

Teacher: And now, we've learned about numbers which can be used in situations which involve an amount and one of two opposite directions. Such numbers are called 'real numbers'.

Bill: Yesterday, we learned how to find the sum of two real numbers. Are we going to learn subtraction of real numbers now?

Teacher: Not right now. First, let's review addition of real numbers.

[The teacher gives the class several exercises in addition of real numbers, which are done mentally.]

Teacher: What is the sum of negative three and positive five?

Tom: Negative two.

Sally: Positive two.

Teacher: Who's right?

Andy: Sally gave the right answer.

Teacher: Why do you think Sally is right?

Andy: Because positive two is the measure of a trip from the starting point of a trip measured by nega-

* This article was written with the co-operation of Miss Eleanor McCoy. Miss McCoy read the preliminary manuscript and made many valuable suggestions for corrections and additions. The section on use of materials in schools is from an unpublished report by Miss McCoy.

tive three to the ending point of a trip measured by positive five.

Ann: Provided the second trip is started at the ending point of the first trip.

Andy: And that's how we agreed we could find sums of real numbers.

Joe: What is positive 5 multiplied by negative 3?

Teacher: What do you think is a simpler numeral than '$^{+}5 \times {}^{-}3$'?

Joe: I don't know. The trips won't work.

Mary: Maybe we should try to find a use for '$^{+}5 \times {}^{-}3$'?

Teacher: We know that real numbers can be used to measure anything which involves an amount of change and one of two opposite directions. What are some such situations?

[Students give several examples.]

Teacher: Let's think for a minute about a pump which can pump water either into or out of a tank, and a camera which takes a movie of the tank while the pump is operating. Suppose the pump works for two minutes at a rate of 5 gallons a minute. What is the change in the volume of water in the tank?

Bill: 10 gallons.

Suzie: But we don't know if it's an increase or decrease.

Tom: Say, real numbers would be handy here. We could use
positive numbers when water flows into the tank
and
negative numbers when water flows out of the tank.

Betsy: Yes, and we could use
positive numbers for an increase in the water in the tank
and
negative numbers for a decrease in the water in the tank.

Teacher: That sounds good. Now, using real numbers as suggested by Tom and by Betsy, what would negative 4 indicate?

Ed: The pump pumping water out at four gallons per minute.

Jim: A decrease of 4 gallons in the tank.

[The teacher asks several questions of this nature.]

Teacher: Now, suppose we used a movie camera and took a motion picture of the tank for 3 minutes while the pump was filling it at the rate of 5 gallons per minute. We could have the film developed and projected on a screen. What change would you *observe on the screen?*

Tom: An increase of 15 gallons, measured by positive 15.

Pete: A decrease of 15 gallons, measured by negative 15.

Sally: Oh, Pete, you're wrong!

Pete: No; see, I imagined the film was run *backward* through the projector.

Sally: Say, here's another use for real numbers:
positive numbers when the film is run forward
and
negative numbers when the film is run backward.

Joe: Positive five times negative three is negative 15.

Pete: And negative five times negative three is *positive* 15.

Sally: No Pete, you're wrong again.

Pete: Well, here's how I figured it out. I figured negative five means the pump was emptying at the rate of five gallons per minute and negative three means the film was running backward, so the change I would see on the screen is an increase of 15 gallons, and we agreed to measure that by positive 15.

Sally: I guess you're right.

[The teacher gives several more problems

to the class and the discussion continues orally.]

Pete: I've discovered a short cut.

Teacher: Fine, Pete, but don't spoil the fun of discovery for the others. Just keep it to yourself. Besides, you might be wrong. It is now near the end of the period. For homework read pages 16 through 20 and write out the exercises on pages 21, 22, and 23 through Part B.

This account of a classroom situation presents in capsule form examples of two of the primary considerations of the UICSM mathematics project: encouragement of discovery by the student, and delay of verbalization of these discoveries. If you should visit the same classroom a month or two later, you would then witness the verbalization of the short cut discovered by Pete. But this verbalization comes only after varied experiences and a thorough treatment of the role variables play in mathematical language.

Objectives of the project

Since the UICSM Project was started in 1951, it has been developing and testing an improved curriculum for college preparatory mathematics (Grades 9–12). At the outset, the staff had three major theses:

1. that a consistent exposition of high school mathematics is possible;
2. that high school students are greatly interested in ideas;
3. that acquiring manipulative skill and understanding basic concepts are complementary activities.[1]

In developing course materials in accord with these propositions, some ideas from contemporary mathematics have been introduced. The classroom experimentation of the past ten years has indicated that contemporary mathematics contains much that is both interesting and valuable to high school students. However, in writing the texts, more concern has been given to consistency than to modernness.

The UICSM staff has been concerned, also, with the importance of keeping the language of textbook and teacher as unambiguous as possible. Perhaps too much was said about this concern in the early years, as there has been a great deal of misunderstanding about the *relative* importance which the UICSM places on the use of language. We *do* consider it important that the language of the textbook (and the teacher) be clear, concise, and unambiguous. After all, a substantial part of mathematics *teaching* consists of instruction in the use of mathematical symbols. If we wish students to understand that there are mathematical ideas *and* that there are symbols for such concepts, we must be sure that our exposition does not deliberately confuse these two things. Please note, however, "precision in exposition is something we expect of the textbook and the teacher, rather than of the learner. Precise communication is a characteristic of a good textbook and a good teacher; correct *action* is a characteristic of the good learner."[2]

Precision in exposition is closely connected with another of our basic considerations, discovery of generalizations by the student. A student entering eighth or ninth grade often thinks of mathematics as a collection of unrelated rules. (It is to be hoped that work being done by such groups as the Madison Project and the University of Illinois Arithmetic Project will improve this situation.) A junior-high student has learned that if he remains quiet in class and does not ask "why-questions," he will be able to do his homework by imitating what the teacher has done in class or by following the rules in

[1] UICSM Project Staff, "The University of Illinois School Mathematics Program," *School Review*, LXV (Winter, 1957), 457–65.

[2] Max Beberman and Herbert Vaughan, *Unit 1 of High School Mathematics* (Teacher's edition) (Urbana, Illinois: University of Illinois Press, 1960), Introduction.

the book. Although this "tell and do" method of teaching has turned out some high-school graduates who appear to be well grounded in mathematics, its effectiveness is limited. Not all problems can be solved by imitation.

This passive attitude of the student must be changed if he is to learn that "mathematics is the classification and study of all possible patterns."[3] So, for example, he must be given opportunities to discover and work with number patterns and to study and classify the number properties exhibited by the patterns. This means, of course, that the textbook must not give the game away. Too many texts have a series of discovery exercises followed immediately by a statement of the thing to be discovered. Thus, there is no real reason for the student to think. In UICSM texts, you will not find rules displayed on pages immediately following discovery exercises. We feel that it is part of the teacher's job to find out whether the student has discovered the correct generalizations. Questions and exercises are included in the text to aid the teacher in this task *without* compelling the student to verbalize his discovery. In fact, premature or incorrect verbalization of a generalization may hinder the use of the generalization.[4,5] Of course, precise verbalization at some time is necessary for purposes of communication and proof, but this verbalization should come only after the individual has become thoroughly familiar with the generalization and has had adequate opportunity to test and refine it.[6] We believe that major emphasis must be placed on the student *doing* mathematics (rather than being told). It is only in this way that he will develop techniques for solving new problems for which there is no teacher to demonstrate solutions. More education and less instruction—more leading out of the student and less putting in to the student—is one of our major aims.

It should be evident from the foregoing remarks that pedagogical considerations are an integral part of the problem of developing an improved mathematics curriculum. And, it was demonstrated in the early years of the UICSM endeavor that high-school teachers need help in understanding both the newer mathematical concepts and the teaching method necessary to make full use of the course content. Accordingly, a program of teacher training was seen as not only desirable, but necessary, if the aim of bringing about improvement in mathematics education was to be achieved. Hence, the training of teachers has been, and continues to be, a major objective of the UICSM Project.

Nature of the work done

The main activities of the Project staff have been the preparation of text materials, the development of teaching methods, and the training of teachers for a new curriculum in college preparatory mathematics for the secondary school.[7]

The Project staff has developed teaching units for Grades 9–12 for the college-capable student. Units now in use or in preparation are:

Unit	*Descriptive title*
1	The arithmetic of the real numbers
2	Pronumerals, generalizations, and algebraic manipulation
3	Equations and inequations; applications
4	Ordered pairs and graphs
5	Relations and functions
6	Geometry
7	Mathematical induction
8	Sequences
9	Exponential and logarithmic functions
10	Circular functions and trigonometry
11	Polynomial functions and complex numbers

[3] W. W. Sawyer, *Prelude to Mathematics* (Harmondsworth, Middlesex, England: Penguin Books, Ltd., 1955), p. 12.

[4] Bert D. Schwartz, "Certain Relations Between Verbalization and Concept Formation: The Destruction of Ideas by Words," Unpublished Ph.D. dissertation, Princeton University, 1948.

[5] Gertrude Hendrix, "Learning by Discovery," The Mathematics Teacher, LIV (May, 1961), 290–99.

[6] *Ibid.*, p. 292.

[7] Support for these undertakings has come from the University of Illinois, the Carnegie Corporation of New York, the U.S. Office of Education, and the National Science Foundation.

Units 1–4 (First year), Units 5–6 (Second year), and Units 7–8 (Third year) are now available from the University of Illinois Press.[8]

The Teacher's edition of each unit consists of the Student's edition, plus Teachers' Commentary pages ("green pages") to be inserted at appropriate places in the Student's edition. The commentary pages contain considerable mathematical background material and teaching suggestions, as well as answers to the problems and exercises in the text.

Since the first "trial classes" in 1952, more than 400 high-school and junior-high-school teachers in *co-operating schools* have been directly involved in the work of the Project. (A co-operating school is one in which the teachers using the program have received special training and submit reports to the Project Center in Urbana. Such schools are visited from time to time by members of the Project staff.) These teachers have contributed suggestions for use in revisions of the texts and Teachers' Commentaries. Also, accounts of their experiences are transmitted to others through the UICSM Newsletter. Some of the participating teachers have conducted in-service courses in UICSM materials, under sponsorship of a school system or university.

The Project staff conducts an annual summer institute in Urbana for the purpose of training teachers in both the content and the pedagogy of UICSM courses. In the summer of 1961 there were 225 teachers enrolled in the Urbana institute, under National Science Foundation sponsorship. Project staff members and teachers from co-operating schools have also taught UICSM courses in other summer institutes.[9]

Use of materials in schools

During the academic year 1960–61, over 28,000 students were using materials prepared by the Project staff. About 12,000 of these students were enrolled in more than 500 classes in co-operating schools. (The other students were in schools using UICSM materials which operate completely independently of the Project Center.)

Units 1–3, although written for use in ninth grade with regular college preparatory classes, are being used in many schools with accelerated classes of eighth graders. It should be pointed out that teachers using Units 1–3 below the ninth-grade level are advised to supplement the text, when it seems necessary, with work on per cent and on intuitive geometry. The Project staff advises schools to make careful selection of students for classes in such an *accelerated* program; we recommend that students chosen have IQ scores about 120 or above and an arithmetic grade achievement score of 8.5 or above. (The latter is essential, since many of the discovery exercises in Units 1–3 presuppose an ability to compute.) A comparison of results on our Unit tests indicates that the eighth-grade students made scores as high as, or higher than, the students in regular ninth-grade classes. Students who study Units 1–3 in the eighth grade usually study Units 4 and 5 in their second year of mathematics; they are then ready for Unit 6—Geometry—at the tenth-grade level.

The UICSM program is a four-year *sequential* program. It is difficult for a student or teacher to begin the use of UICSM materials at any place other than Unit 1. It is equally difficult to attempt to

[8] Orders may be sent to the University of Illinois Press, Urbana, Illinois. Titles are of the form:
UICSM. Unit——of HIGH SCHOOL MATHEMATICS————Edition
Students' editions (paperbound) are sold at these prices: Units 1, 2, 3, 4–$3.00 per set (Separate units–$1.00 each); Unit 5–$1.50; Unit 6–$2.00; Unit 7–$1.25; Unit 8–$1.75.
Teachers' editions (looseleaf) are priced as follows: Units 1, 2, 3, 4–$6.00 per set (*not* sold separately); Unit 5–$3.00; Unit 6–$4.00; Unit 7–$2.75; Unit 8–$4.00.

[9] In 1961, such institutes were held at University of Arizona, University of Connecticut, University of Hawaii, University of Kansas, Rhode Island College of Education, Sacramento State College, Wayne State University, and Western Washington College of Education.

use any single unit to supplement other materials being used in a school.[10]

It is possible for an experienced teacher, that is, one who has taught UICSM Units 1–4 for at least two years, to take a class of sophomores who have not studied Units 1–4 and, after some transition work, use Unit 5 or Unit 6 with these students.

Future plans

As new programs now being developed for the elementary and junior-high grades become more widely used, some of the material now in our earlier units should become unnecessary. We feel that this is a mark of progress in mathematics education in our schools. We, and other groups, will then be able to build an even better curriculum for the secondary school.

To meet the growing demand for people trained to do the kind of teaching required for the UICSM program, improved methods for teacher training must be developed. We believe that part of such training must be the observation of a class of students taught by a teacher skilled in the techniques of discovery teaching. This has been partially accomplished by using demonstration classes at summer and in-service training conferences. One difficulty with this approach is that demonstration classes are not always available during summer sessions or on Saturday mornings. For this reason, we have produced a series of training films showing a beginning UICSM class in action. The films have been designed to illustrate pedagogical techniques, rather than to teach subject matter. Plans for the distribution of the films are now being made.[11]

[10] However, it is possible for students who have completed one or more years of study of the UICSM curriculum to change to some other textbook. Several students who have moved away from schools using our materials have done this without any hardship. An analysis of the UICSM textbooks will show that they contain most of the traditional topics together with some of the newer topics recommended by the Commission on Mathematics of the CEEB.

[11] Inquiries concerning the UICSM film series should be addressed to Professor Gertrude Hendrix, UICSM, 1208 West Springfield, Urbana, Illinois.

In assessing the current situation in the schools of the United States, we feel that it is clear that the UICSM along with other groups have exerted a profound influence on the mathematics curriculum. We think the nation's schools are ready to move in larger numbers toward curriculum improvement in mathematics. But, the problem of bringing about significant and widespread changes is still far from solution. Many teachers are most reluctant to undertake teaching the newer courses. From our experience, we have found that, in installing new curriculum materials in our schools, teachers need training and supervision. And, this need is present even with materials whose teachability has been perfected over many years of careful development.

Because we are convinced beyond doubt of the desirability (we might even say, *necessity*) of giving practical, realistic training to teachers who are interested in using the newer curriculum materials in mathematics, we have formulated plans for such training. We hope to continue our summer institute programs and, as a concomitant part of the training program, maintain a supervisory follow-up during the academic year following each institute. Such supervision would, as in the past, include visits to co-operating schools by staff members for the purpose of assisting teachers (through conferences and demonstration teaching) and informing school patrons of the changes taking place, assuring them of the care with which these changes are being made.

Bibliography

Beberman, Max, *An Emerging Program of Secondary School Mathematics.* Inglis Lecture Series. Cambridge: Harvard University Press, 1958.

Beberman, Max, "Improving High School Mathematics Teaching," *Educational Leadership* (December, 1959).

Hendrix, Gertrude, "Variable Paradox—A Dialog in One Act," *School Science and Mathematics* (June, 1959).

Hendrix, Gertrude, "Learning by Discovery," The Mathematics Teacher, LIV (May, 1961), 290–99.

HENDRIX, GERTRUDE, and SIMS, BYRL, "The UICSM Teacher Training Films," *American Mathematical Monthly* (August-September, 1960).

MCCOY, M. ELEANOR, "A Secondary School Mathematics Program," *The Bulletin of the National Association of Secondary School Principals* (May, 1959).

UICSM STAFF, "The University of Illinois Mathematics Program," *The School Review*, (Winter, 1957).

UICSM STAFF, "Words, 'Words', "Words", " THE MATHEMATICS TEACHER, L (March, 1957), 194–98.

UICSM STAFF, "Arithmetic With Frames," THE ARITHMETIC TEACHER (April, 1957).

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least two months prior to the month in which the issue appears to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

FORTIETH ANNUAL MEETING

April 16–18, 1962
Jack Tar Hotel, San Francisco, California
Kenneth C. Skeen, 3355 Cowell Road, Concord, California

JOINT MEETING WITH NEA

July 4, 1962
Denver, Colorado
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

TWENTY-SECOND SUMMER MEETING

August 23–25, 1962
University of Wisconsin, Madison, Wisconsin
Don McCloskey, Wisconsin High School, Madison, Wisconsin
Stephen S. Willoughby, School of Education, University of Wisconsin, Madison 6, Wisconsin

Other professional dates

The Association of Teachers of Mathematics in New England
December 9, 1961
Wentworth Institute, Boston, Massachusetts
Barbara B. Betts, D. C. Heath & Company, Boston 16, Massachusetts

Chicago Elementary Teachers Mathematics Club
December 11, 1961
Private Dining Room, Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Mildred C. Rogers, Warren Elementary School, 9210 South Chappel Avenue, Chicago 17, Illinois

Men's Mathematics Club of Chicago and Metropolitan Area
December 15, 1961
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Charles W. Moran, 2950 Jarlath Street, Chicago 45, Illinois

Men's Mathematics Club of Chicago and Metropolitan Area
January 19, 1962
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Charles W. Moran, 2950 Jarlath Street, Chicago 45, Illinois

Association of Mathematics Teachers of New Jersey, Sectional Meeting
January 20, 1962
Jersey City State College, Jersey City, New Jersey
Prof. Ernest R. Duncan, Jersey City State College, Jersey City, New Jersey

New York Society for the Experimental Study of Education; Section 10—Mathematics
January 27, 1962
256 Thompson Hall, Teachers College, Columbia University, New York City
Mary G. Rule, 58 Spring Avenue, Bergenfield, New Jersey

The Mathematics Club of Greater Cincinnati
February 1, 1962
Shroder Junior High School, 3500 Lumford Place, Cincinnati, Ohio
Roy D. Matthews, Board of Education, 608 East McMillan Street, Cincinnati 6, Ohio

Association of Mathematics Teachers of New Jersey, Winter Meeting
February 3, 1962
Paramus High School, Paramus, New Jersey
Prof. Ernest R. Duncan, Jersey City State College, Jersey City, New Jersey

Women's Mathematics Club of Chicago and Vicinity
February 3, 1962
English Room, Marshall Field & Company, Chicago, Illinois
Dr. Ruth Ballard, University of Illinois, Navy Pier, Chicago 11, Illinois

Edited by Howard Eves, University of Maine, Orono, Maine

Some reflections on Gulliver's Travels

by C. F. Merrill, University of Cincinnati, Cincinnati, Ohio

Introduction

It has been stated that *"Gulliver's Travels* draws upon at least five traditions of world literature . . . , the literal travel account, realistic fiction, utopian fiction, symbolism, and the fantastic voyage [1]."* There is, however, another aspect of *Gulliver's Travels* which perhaps some of its readers have not considered.

The mathematics employed by Swift in giving his book some of its brilliantly satirical and imaginative tone is an aspect which might merit study. Only three approaches will be considered here, though there are others which a reader may be interested in exploring on his own. Considered here are the mathematical proportions used by Swift, his gibes at mathematics and mathematicians, and some of the departures from physical and biological reality of which he has been guilty.

The mathematical proportions

In the first part of the book, Lemuel Gulliver, then a ship's surgeon, is shipwrecked and washed up on a small island of Lilliput which is inhabited by tiny people six inches high. Here Swift has created a world in miniature. Following Gulliver through his many adventures in this country, one wonders how Swift arrived at the size of these tiny people, and whether he is consistent in assigning various sizes to the objects in the environment. Upon inspection, it is found that Swift has used a proportion of *twelve* to *one*, and that he has held rather well to this proportion throughout. To illustrate, consider the following quotations which show his proportions and which, at the same time, exhibit consistency. Upon his first view of one of the Lilliputians, Gulliver says, "I perceived it to be a human creature not six inches high." In conveying Gulliver, prostrate upon a wagon, from the beach where he was washed up to his living quarters, the Lilliputians used "fifteen hundred of the Emperor's largest horses, each about four inches and a half high." When it was decided by a Lilliputian Council that Gulliver should live, the problem of supplying him with food and drink was encountered. Thus, "his Majesty's mathematicians, having taken the height of my body by the help of a quadrant, and finding it to exceed theirs by the proportion of twelve to one, they concluded from the similarity of their bodies that mine must contain at least 1728 of theirs." Later, in writing of the manners and customs of the Lilliputians, he states, "as the common size of the natives is somewhat under six inches high, so there is an exact proportion in all other animals, as well as plants and trees: for instance the tallest horses and oxen are between four and five inches in height, the sheep one inch and a half, more or less; their geese about the largeness of a sparrow, a lark the size of a common fly, trees about seven feet high." The Lilliputians used a rule of one inch long.

* Numbers in brackets refer to the Notes at the end of the article.

During Gulliver's residence in Lilliput of nine months and thirteen days, he became in need of new clothing. He gives the following account of their construction. "The sempstresses took my measure as I lay on the ground, one standing at my neck, and another at my mid-leg, with a strong cord extended, that each held by the end, while the third measured the length of the cord with a rule of an inch long. Then they measured my right thumb, and desired no more; for by a mathematical computation, that twice round the thumb is once around the wrist, and so on to the neck and the waist, ..."

Later we find Gulliver in the land of the giants, Brobdingnag. Again we observe the same proportion of twelve to one, in a reversed direction, and again find them quite constant. Upon first view of one of the giants, Gulliver exclaims, "He appeared as tall as an ordinary spire-steeple and took about ten yards at every stride . . . ." If we translate the giant's ten-yard stride into one of ordinary size using the proportion of twelve to one, it becomes a thirty-inch stride, roughly that of an average man. In a later chapter, Gulliver fights and kills a great rat and measures its tail, finding it to be six feet long. Converted to ordinary size this becomes six inches long.

He is taken to the metropolis by the giant who had found him, where he relates, "my master hired the crier to give notice of a strange creature . . . not so big as a splackmuck, and in every part of the body resembling a human creature."

Although Swift apparently intended the proportion to be twelve to one, he never definitely states the fact, but most nearly reveals it in this statement: "a hail stone is near eighteen hundred times as large as one in Europe." If the hailstone were *exactly* 1,728 times as large as one in Europe, then the proportion would be *exactly* twelve to one; however 1,728 is very "near eighteen hundred."

Gulliver leaves Brobdingnag and before long is once again standing on some unknown island. Swift saw fit to send Gulliver on several more equally interesting voyages. Whereas Swift was concerned with the extension and reduction in physical size in Gulliver's first two voyages, in later voyages he is concerned with the extension of more abstract things, such as the extension of human understanding in the land of the Houyhnhnms and the extension of life in Luggnogg.

The gibes at mathematicians

Just why Swift made so much fun of mathematicians and natural scientists in the later voyages is not certain. Ingenious scholars have traced almost everything in *A Voyage to Laputa* to contemporary science and to certain prevalent popular reactions to scientific doctrines and discoveries. It would seem that here Swift is aiming at pedantry and ostentation in science and learning [2]. It is known that Swift had little appreciation of the purer and less practical aspects of mathematics and science.

The following quotations from *A Voyage to Laputa* illustrate Swift's attitude. "His Majesty took not the least notice of us, although our entrance was not without sufficient noise, by the concourse of all persons belonging to the court. But he was then deep in a problem, and we attended at least an hour, before he could solve it. There stood by him on each side a young page, with flaps in their hands, and when they saw he was at leisure, one of them gently struck his mouth, and the other his right ear; at which he started like one awaked on the sudden, ..." "My dinner was brought, and four persons of quality . . . did me the honour to dine with me. We had two courses of three dishes each. In the first course there was a shoulder of mutton, cut into an equilateral triangle, a piece of beef into a rhomboides, and a pudding into a cycloid. . . . The servants cut our bread into cones, cylinders, parallelograms, and several other mathematical figures." "Those to whom the King had entrusted me, observing how

ill I was clad, ordered a tailor to come next morning, and take my measure for a suit of clothes. This operator did his office after a different manner from those of his trade in Europe. He first took my altitude by a quadrant, and then with a rule and compasses described the dimensions and outlines of my whole body, all of which he entered upon paper, and in six days brought my clothes very ill made, and quite out of shape, by happening to mistake a figure in the calculation. But my comfort was, that I observed such accidents very frequent, and little regarded." "Their houses are very ill built, the walls bevil, without one right angle in any apartment, and this defect ariseth from the contempt they bear to practical geometry, which they despise as vulgar and mechanic, ... And although they are dexterous enough upon a piece of paper in the management of the rule, the pencil, and the divider, yet in the common actions and behavior of life, I have not seen a more clumsy, awkward, and unhandy people, nor so slow and perplexed in their conceptions upon all other subjects, except those of mathematics and music."

Later Gulliver inspects a mathematical school in Laputa, where he sees the master teach his pupils "after a method scarce imaginable to us in Europe. The proposition and demonstration were fairly written on a thin wafer, with ink composed of a cephalic tincture. This the student was to swallow upon a fasting stomach, and for three days following eat nothing but bread and water. As the water digested, the tincture mounted to his brain, bearing the proposition along with it." But for a number of reasons the method seemed to fail.

R. Whately has given one explanation of Swift's attitude toward mathematics and mathematicians: "It is worth observing that some of those who disparage some branch of study in which they are deficient, will often affect more contempt for it than they really feel. And not unfrequently they will take pains to have it thought that they are themselves well versed in it, or that they easily might be, if they thought it worth while;—in short, that it is not from hanging too high that the grapes are called sour. Thus Swift, in the person of Gulliver, represents himself, while deriding the extravagant passion for Mathematics among the Laputians, as being a good mathematician. Yet he betrays his utter ignorance, by speaking 'of a pudding in the *form of a cycloid*' evidently taking the cycloid for a *figure*, instead of a *line*. This may help to explain the difficulty he is said to have had in obtaining his Degree [3]."

And Alfred North Whitehead has written: "(Gulliver) describes the mathematicians of (Laputa) as silly and useless dreamers, whose attention has to be awakened by flappers. Also, the mathematical tailor measures his height by a quadrant, and deduces his other dimensions by a rule and compasses, producing a suit of very ill-fitting clothes. On the other hand, the mathematicians of Laputa, by their marvellous invention of the magnetic island floating in the air, ruled the country and maintained their ascendency over their subjects. Swift, indeed, lived at a time peculiarly unsuited for gibes at contemporary mathematicians. Newton's *Principia* had just been written, one of the great forces which have transformed the modern world. Swift might just as well have laughed at an earthquake [4]."

The departures from scientific reality

Let us now consider some scientific inconsistencies which Swift incorporated into his work. The difficult part of examining this aspect of *Gulliver's Travels* is determining at what point Swift knowingly departs from the truth, and at what points he is ignorant of the actual principles involved.

Although Swift needed to employ changes in physical size to make his work the highly imaginative and satirical prod-

uct that it is, many of his changes were biologically impossible. J. B. S. Haldane, a noted biologist, says: "It is easy to show that a hare could not be as large as a hippopotamus, or a whale as small as a herring. For every type of animal there is a most convenient size, and a large change in size inevitably carries with it a change in form [5]." This implies that there could not be a land of tiny people or a land of giant people "in every part of their body resembling a human creature."

Let us look at a few examples where, in keeping with the physical proportions, Swift has violated certain physical and biological realities. In Lilliput, "The Emperor had a mind one day to entertain me with several of the country shows . . . . I was diverted with none so much as that of the rope-dancers, performed upon a slender white thread . . . twelve inches from the ground. . . . These diversions are often attended with fatal accidents . . . . I myself have seen two or three candidates break a limb." This demonstrates ignorance of principles. A biologist knows that it is highly improbable that a creature six inches high could break a limb by falling one foot, for "to a mouse and any smaller animal gravity presents practically no danger. You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away. . . . For the resistance presented to movement by the air is proportional to the surface of the moving object. Divide an animal's length, breadth, and height each by ten; its weight is reduced to a thousandth, but its surface only a hundredth. So the resistance to falling in the case of the small animal is relatively ten times greater than the driving force [6]."

One might say that perhaps Swift has kept to the same proportion in respect to gravity also. In fact, he states that nature observes the same proportion in "all her operations." This would mean that the pull of gravity would be twelve times greater, which, if it didn't kill the tiny people, would render Gulliver totally immovable.

Swift again violates known principles in discussing the eyesight of the people of Lilliput. He acknowledges in the following passage that the eyesight of these tiny people is much sharper than that of his own people. If these people enjoy the same proportions as do his people, this is impossible. In order that smaller animals have better sight, they need bigger eyes. Consider the following: "The Emperor was amazed at the continual noise (my watch) made, and the motion of the minute hand, which he could easily discern; for (the Lilliputians') sight is much more acute than ours." But: "Acute sight is dependent upon size and number of the rods and cones in the back of the aninal's eyes. Thus, in order that they should be of any use at all, the eyes of smaller animals have to be much larger in proportion to their bodies than our own [7]."

In determining how much food was necessary to support Gulliver in Lilliput, the Emperor's mathematicians used a twelve to one proportion and concluded that Gulliver would eat as much as would 1,728 Lilliputians. This is improbable, for, to again quote Haldane, "All warm-blooded animals at rest must lose the same amount of heat from a unit area of skin, for which purpose they need a food-supply proportional to their *surface* and not to their *weight*. Five thousand mice weigh as much as a man. Their combined surface and food or oxygen consumption are about seventeen times a man's. In fact a mouse eats about one quarter of its own weight of food every day, which is mainly used in keeping it warm [8]." From this statement we may conclude that the Lilliputians ate far more in proportion to their bodies than would Gulliver. In fact it would seem that if their metabolic rate was the same as ours, then they would need to eat nearly all the time just to function correctly. On the other hand, if their metabolism was also on a twelve to one proportion, then it seems probable that they

would be as starfish and have little movement.

In creating the giants of Brobdingnag, Swift has also made a few biological departures from fact. We find the same departure in *Pilgrim's Progress.* "Let us . . . consider a giant man sixty feet high—about the height of Giant Pope and Giant Pagan in the illustrated *Pilgrim's Progress* . . . . These monsters were not only ten times as high as Christian, but ten times as wide and ten times as thick, so that their total weight was a thousand times his, or about eighty or ninety tons. Unfortunately the cross sections of their bones were only a hundred times those of Christian, so that every square inch of giant bone had to support ten times the weight borne by a square inch of human bone. As the human thigh-bone breaks under about ten times the human weight, Pope and Pagan would have broken their thighs every time they took a step [9]." Since the giants of Brobdingnag were *twelve* times the size of Gulliver, they would not have been able to stand without breaking their legs.

Gulliver also notes of Brobdingnag, "The kingdom is much pested with flies in summer; and these odious insects, each of them as large as a Dunstable lark, hardly gave me any rest while I sat at dinner." Elsewhere he is attacked by wasps "as large as partridges." Again, the size of these insects is a biological impossibility. For "insects take air directly to every part of their body by tiny blind tubes called tracheae which open to the surface at many different points. Now, although by their breathing movements they can renew the air in the outer part of the tracheal system, the oxygen has to penetrate the finer branches by means of diffusion. Gases can diffuse easily through very small distances, not many times larger than the average length travelled by a gas molecule between collisions with other molecules. But when such vast journeys—from the point of view of a molecule—as a quarter of an inch have to be made, the process becomes slow. So the portions of an insect's body more than a quarter of an inch from the air would always be short of oxygen. In consequence hardly any insects are much more than half an inch thick [10]." This renders insects as large as larks and partridges absurd.

Gulliver escapes from the peninsula of Brobdingnag by being carried high above the ocean by a huge bird. This also is biologically impossible. To verify this, consider the following quote: "An angel whose muscles developed no more power weight for weight than those of an eagle or a pigeon would require a breast projecting for about four feet to house the muscles engaged in working its wings, while to economize in weight, its legs would have to be reduced to mere stilts. Actually a large bird such as an eagle or kite does not keep in the air mainly by moving its wings. It is generally to be seen soaring, that is to say balanced on a rising column of air. And even soaring becomes more and more difficult with increasing size. Were this not the case eagles might be as large as tigers . . . [11]."

Gulliver's next voyage takes him to the flying island of Laputa. This island is made to fly by a huge loadstone (magnet) fixed and easily rotated on an axle. He explains its function thus: "The stone is anchored at one of its sides with an attractive power, and at the other with a repulsive. Upon placing the magnet erect with its attractive end towards the earth, the island descends, but when the repelling extremity points downward, the island mounts upwards. When the position of the stone is oblique, the motion of the island is so too. For in this magnet the forces always act in lines parallel to its direction. . . . When the stone is put parallel to the plane of the horizon, the island standeth still. For in that case the extremities of it being at equal distances from the earth, act with equal force, the one in drawing downwards, the other in pushing upwards, and consequently no motion can ensue."

Difficulties seem to appear here also. For if the attracting end of the magnet were pointed toward the earth, the island would fall with an acceleration greater than that of gravity and would be difficult to control. The operators of the magnet would be in free fall and would have some difficulty in rearranging the magnet to slow the island in the fall. In the case where the loadstone is parallel to the earth, we know that the forces acting on the stone as *described* would produce rotation of either the island or the magnet to the position where the attracting end is pointed downwards.

Conclusion

It has certainly not been the author's intention to infer through various criticisms of *Gulliver's Travels* that it is an inferior book; on the contrary, it is a masterpiece of satire exhibiting vivid imagination throughout. The major significance of the book lies in these satirical observations of the English culture of Swift's time.

One can justify bringing out these various discrepancies by saying that they are interesting. In reading any story it is interesting to watch for places where the author's imagination has led him to stray from the actual existing principles. Of course, this in no way lessens one's respect for the book.

Notes

1. Harnstien, Percy, and Brown, *The Reader's Comparison to World Literature* (New York: Dryden Press, 1956), p. 196.
2. J. R. Newman, "Commentary on the Island of Laputa," *The World of Mathematics* (New York: Simon and Schuster, Inc., 1956), IV, 2210–13.
3. R. Whately, *Annotations to Bacon's Essays, Essay L*, as quoted in R. E. Moritz, *Memorabilia Mathematica* (New York: Macmillan Co., 1914), p. 372.
4. A. N. Whitehead, *An Introduction to Mathematics* (New York: Oxford University Press, 1948), p. 3.
5. J. B. S. Haldane, "On Being the Right Size," *The World of Mathematics* (New York: Simon and Schuster, Inc., 1956), II, 952.
6. *Ibid.*, p. 953.
7. *Ibid.*, p. 956.
8. *Ibid.*, p. 955.
9. *Ibid.*, p. 952.
10. *Ibid.*, p. 954.
11. *Ibid.*, p. 956.

The astonishing prediction in Gulliver's Travels

by Howard Eves, University of Maine, Orono, Maine

In *A Voyage to Laputa*, Dean Swift has Gulliver report on some of the remarkable astronomical achievements of the Laputians. Gulliver says, in part, "(The Laputian astronomers) spend the greatest part of their time in observing the celestial bodies, which they do by the assistance of glasses far excelling ours in goodness. . . . This advantage hath enabled them to extend their discoveries much further than our astronomers in Europe; for they have made a catalogue of ten thousand fixed stars, whereas the largest of ours do not contain above one third part of that number. They have likewise discovered two lesser stars, or satellites, which revolve about Mars, whereof the innermost is distant from the center of the primary planet exactly three of his diameters, and the outermost five; the former revolves in the

space of ten hours, and the latter in twenty-one and an half; so that the squares of their periodical times are very near in the same proportion with the cubes of their distances from the center of Mars, which evidently shows them to be governed by the same law of gravitation, that influences the other heavenly bodies."

Before going on, let it be noted that *Gulliver's Travels* was originally published in 1726.

In 1877, a good 150 years after the publication of *Gulliver's Travels,* the American astronomer Asaph Hall, using the finest telescope of the time, the 26-in. refractor at the Naval Observatory in Washington, D.C., discovered that Mars possesses two small satellites or moons. These satellites, which have diameters probably under ten miles, are so near the planet Mars that they can be seen only with sufficiently large telescopes, and even then only at favorable times. Suitable telescopes for this observation were not made until about a century after the publication of *Gulliver's Travels.*

Asaph Hall named the two satellites Phobos and Deimos. Phobos, which is the inner satellite, completes a sidereal revolution in 7 hours and 39 minutes, and revolves about Mars at a distance of about 5,800 miles from the center, or about 3,700 miles from the surface, of the planet. The sidereal period of Deimos is 30 hours and 18 minutes and revolves at about 14,600 miles from the center of Mars. Deimos is smaller than Phobos and is only about one-third as bright.

Of the above surprising coincidences, Charles P. Olivier (former director of the Flower and Cook Astronomical Observatories at the University of Pennsylvania) has written, "When it is noted how very close Swift came to the truth, not only in merely predicting two small moons, but also the salient features of their orbits, there seems little doubt that this is the most astonishing 'prophecy' of the past thousand years as to whose full authenticity there is no shadow of doubt. . . . (Phobos') period of less than eight hours obliges it to rise in the west and set in the east. In this . . . respect it is unique among all bodies in the universe, so far discovered. Yet Swift had this fact also included."*

The remarkableness of it all does not quite end here, for these same two satellites were also mentioned by Voltaire (1694–1778) in his story of *Micromegas!*

* "Mars," *Encyclopedia Americana* (1956 ed.).

Have you read?

STRINGFELLOW, THOMAS L., and STRINGFELLOW, EMMA L., "A Brief History of Statistics: A Contribution for the Enrichment of High School Mathematics," *School Science and Mathematics,* January, 1961, pp. 1–4.

The question is shall we teach statistics in the secondary schools; and if so, what are the objectives to be attained? The authors feel that mathematics students are intrigued by statistics, and as schools introduce modern mathematics (which often includes statistics), the student is surprised to find statistics dating from 3000 B.C. For example, "statistics" is derived from *status,* the Italian word for state. The influence of probability on statistics dates from the seventeenth century with Pascal, Fermat, and Huygens directing their attention to it.

From there we come to De Moivre with his "normal curve" and average. These topics were further studied by Gauss and Laplace in 1775, bringing out the "probability of error" and "standard deviation." However, it was left for Galton and Pearson to bring forth "percentiles," "regression," "Chi square," and "correlation coefficient."

Your students may be surprised to know that statistics was first offered as a college course in Columbia University in 1880. If you plan to do any statistics in your school, you should read this.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

RESPONSES

Note: It is the policy of THE MATHEMATICS TEACHER to publish responses to reviews which appear in the magazine when the writer and editors consider the additional information contributes to a considered evaluation of the book reviewed. Miss Henrietta O. Midonick's response which follows provides additional facts and judgments about *The History of Mathematics* by Joseph E. Hofmann.

My attention has been called to a review (March, 1961) of *Classical Mathematics* by Joseph E. Hofmann (Philosophical Library, 1959), a translation by Henrietta O. Midonick of Dr. Hofmann's *Geschichte der Mathematik*, Bde. II, III. This review, written by Dr. Vera Sanford, contains a reference to another review by Dr. Sanford (March, 1958) of *The History of Mathematics* by Joseph E. Hofmann (Philosophical Library, 1957), a translation by Frank Gaynor and Henrietta O. Midonick of Dr. Hofmann's *Geschichte der Mathematik, erster Teil.* Inasmuch as these reviews contain a number of errors, some of which may be considered serious, and certain conclusions which, in my opinion, are unjustified, I feel that it is incumbent upon me to answer these criticisms.

The March, 1958 review contains the reviewer's conjecture, "It is likely, though this is not stated, that this translation is considerably abridged from the original text." The conjecture is followed in the March, 1961 review by the positive assertion, "This book and its predecessor . . . are abridged translations of Professor Hofmann's three-volume *Geschichte der Mathematik*. . . . " Both the conjecture and the assertion are wrong. The translations are complete and faithful renderings of the entire text of Professor Hofmann's work. Moreover, the translators were at great pains at every point to keep to the spirit and intent of the author as well as to the content.

Let me illustrate by discussing a specific criticism in the 1958 review: "The word *Contingenzwinkel* [the reviewer's spelling, not Professor Hofmann's] is translated 'angle of contingency' [the translation has angle of contingence, not angle of contingency], where the English-American expression 'angle of contact' or 'horn angle' might be familiar to some, but where a description of the angle would have made the matter clear." I will restrict my reply to two main points.

First: The explanation which the reviewer quite properly considers desirable is, in fact, clearly stated in the translation (see pp. 22 and 60 of *The History of Mathematics*, 1957). I find it amazing that the reviewer who speaks of a "first reading" of the book and also of a "later reading" should have failed completely to notice both printed explanations in the translation.

Second: Let us consider the choice of words for the translation of *Kontingenzwinkel*. Many writers since the days of Democritus have devised terms to designate this "angle". Among these terms we find "angle of contact", "mixed angle", "horn-like (or hornlike) angle", "cornicular angle", "angle of contingence". None of these terms can be relied upon, without additional description or explanation, to evoke the same image in the minds of all. Hence, since an explanation is necessary (and contrary to the statement in the review, the German text and the translation each supplied an explanation, not once but twice), the choice of a term should be made on the basis of the greatest potentiality for communication as well as on lexicographical availability and usage. The reviewer states, "The English-American expression 'angle of contact' or 'horn angle' might be familiar to some . . . ". That is to say, familiarity with these terms, if it exists, is limited. Why is it limited? One reason for this, I suggest, is that the terms preferred by the reviewer may not be the best ones for the purpose. Granting that familiarity through usage is a strong argument in favor of employing a given word, this advantage vanishes when the familiarity is doubtful and limited only to "some". At any rate, as regards the great majority of readers, it is clear that all these terms are on a fairly even footing of strangeness, so that, for these readers, familiarity is not a factor to be considered. However, we must be careful to guard against half-familiar terms which may prove to be misleading, as I shall indicate later. Murray's *A New English Dictionary on Historical Principles*, Vol. II, gives as the first definition of "angle of contingence", "The infinitesimal angle between the circumference of a circle and its tangent." The reviewer will surely recognize the *English Dictionary* as an authoritative source of "English-American" words and usage. *Encyclopaedia Britannica* (1957), Vol. VII, p. 340, states, "This great work, conceived 70 years before its completion in 1928 . . . involved years of research by hundreds of volunteers on both sides of the Atlantic. Editor in Chief James A. H. Murray

stated that most of the cooperation from 'men of Academic standing' came from 'Professors in American Universities' and their students." The first definition of "angle of contingence" given by the *English Dictionary*, as quoted above, is entirely in agreement with Dr. Hofmann's explanation of *Kontingenzwinkel* and "angle of contingence" is most acceptable as a translation of the German term.

On the other hand, I must emphatically reject Dr. Sanford's suggestion that "horn angle" be used. Indeed, if this suggestion should be followed, and if at the same time we should accept the suggestion made in the March, 1961 review that "one would do well to keep Dr. Boyer's *History of Analytic Geometry* and his *Concepts of the Calculus* close at hand" while reading Professor Hofmann's work, then we should really be in trouble. On page 22 of *The Concepts of the Calculus*, Dr. Boyer describes "horn angles" as "the angles formed by curves which have a common tangent at a point." On page 173 of the same work he confirms this definition: "the angle of contact (horn angle) formed by two curves with a common tangent . . . ." The "angle of contact" and "horn angle" are thus taken here as interchangeable terms. Now the difficulty that arises is that the concept Dr. Boyer ascribes to these terms is not the same as the concept which Dr. Hofmann ascribes to *Kontingenzwinkel*, the angle formed by an arc of a circle and its tangent. Obviously, if we accept Dr. Boyer's definition of an "angle of contact" or "horn angle", these terms are definitely ruled out as possible translations of *Kontingenzwinkel*. Moreover, to point up the confusion existing in the usage of these terms in current American literature, I will refer to Dr. Sanford's own work, *A Short History of Mathematics*, where Dr. Sanford gives still another explanation of the term "horn angle". The concept which she ascribes to "horn angle" differs from the concept ascribed to the same term by Dr. Boyer. It also differs from the concept ascribed to "angle of contingence" by the definition in Murray's *English Dictionary;* but most important of all, in connection with the translation of Dr. Hofmann's work, it differs from the concept ascribed by him to *Kontingenzwinkel*. Dr. Sanford, in discussing Euclid, Book III, states:

> It is in this Book that Euclid defines an angle of a segment as that angle "contained by a straight line and a circumference of a circle, the so-called "horn angle" . . . (V. Sanford, *A Short History of Mathematics*, 1930, pp. 272–3).

Dr. Sanford gives no citation for her quotation. In the volume available to me, undoubtedly because of a printer's error, it is impossible to tell where the quotation beginning "contained . . ." comes to an end. Since no additional explanation is given by Dr. Sanford, some readers might well wonder about the position of the straight line mentioned. For a clear delineation of the angle Dr. Sanford refers to, we may turn to T. C. Heath, *The Thirteen Books of Euclid's Elements*, Vol. I, p. 253, where the discussion and accompanying diagram indicate clearly that the angle of a segment (the emphasis is on the word *of*) is a "mixed" angle between a chord and a circle and the circumference of the smaller segment cut off by it. The plain fact is that the angle which Dr. Sanford describes as and names "horn angle" is not Dr. Hofmann's *Kontingenzwinkel* (*zwischen Bogen und Tangente*) and, again, I must conclude that "angle of contingence" is by far the best and most accurate choice.

Considerations such as these make translation an extremely delicate work. It goes without saying that translation requires a great deal more of research than can ever be apparent from an inspection of the finished work.

Now let us turn to the word "applicate" to which Dr. Sanford has expressed her objection. In describing the procedure employed by Descartes in *La Géométrie* to solve the Pappus problem, Dr. Hoffmann used the term "*Applikate*" to denote a line $AP=y$. To have translated this word as "ordinate" as the reviewer suggests would have been easy enough. But this would have been an error in translation. The German equivalent for the modern term is beyond any shadow of a doubt well known to Dr. Hofmann. Hence, his choosing "Applikate" could not have been otherwise than purposeful. If the translator had used the modern word "ordinate" for the old word "Applikate", Dr. Hofmann's purpose in the use of "Applikate" would have been defeated. Moreover, it would have had the effect of reinforcing a widely held misconception that Descartes was making use of "Cartesian axes" as we known them today. This, of course, was not the case. See D. J. Struik, *A Concise History of Mathematics* (1948), Vol. II, p. 134, and J. F. Scott, *The Scientific Work of René Descartes* (London, 1952), pp. 94 and 99. David Eugene Smith and Marcia L. Latham lay a careful foundation of historical references before using the word "ordinate" in their translation of *La Géométrie* and even after this has been done, they surround it with parentheses which are not in the original French (*The Geometry of René Descartes*, pp. 67, 84, and others). Dr. Hofmann, by the use of quotation marks, reminds us that we are not speaking of analytical geometry as we study it today, but rather of the origin and early stages of its development. Having made this point by introducing the old word "Applikate" between quotation marks, Dr. Hofmann later gives up the use of quotation marks, and the translator follows this very procedure with the old word "applicate."

As to the reviewer's pungently expressed view that the polyhedron theorem, given by Dr. Hofmann *in symbolical* form, should have been translated, her reason being that she finds it possible to attach verbal meanings in German to certain letters used, it appears to me that this

view would have been more to be expected if we were back in the days of "syncopated algebra". The art of mathematical symbolism is not so restricted today. To show how easy it is to give meaning to symbols, consonant with the concepts involved in the theorem, without recourse to the German language, let me quote from Felix Klein's *Elementary Mathematics from an Advanced Standpoint, Geometry*, (translation by E. R. Hedrick and C. A. Noble) p. 108: "Euler observed that if an ordinary polyhedron has E corners, K edges, and F faces *we always have the relation* $E+F=K+2$." These are the very symbols which were carried over from Dr. Hofmann's work to the translation.

Thus, neither the polyhedron theorem nor the use of the word "applicate" could have introduced a "red herring" into the work. The translation does not, in fact, offer a "red herring" here or anywhere else in the book, but on the contrary, proceeds as does Dr. Hofmann, with clarity and with due respect for the facts.

Regarding a complex of criticisms relating to titles and subtitles, I will say only that Dr. Sanford does not make it clear that the first subtitle mentioned by her is the subtitle which clearly delimits the scope of Volume I, and that this subtitle was printed in the translation (*The History of Mathematics*, 1957, page preceding Chapter 1). If the reviewer inadvertently overlooked the translation of this subtitle, then the criticism of "misleading generality", however unjustified in actuality, is understandable. If, however, she did take note of the translation of the subtitle, I fail to see how this criticism can be directed to the translation.

The merit of Dr. Hofmann's work, apart from the bibliography appended, was specifically noted by Dr. Boyer in his review of Dr. Hofmann's *Geschichte der Mathematik, erster Teil* (*ISIS*, March, 1955), "These lists alone would justify the appearance of the book, which needs no justification beyond the pleasure it will bring to students of history and mathematics."

The text of Dr. Hofmann's history has its own intrinsic merit, charm, and authority, and adherence to the original text has been a guiding principle for the translator.—*Henrietta O. Midonick, 40 East 9th St., New York 3, New York.*

BOOKS

Algebra I, Charles F. Brumfiel, Robert E. Eicholz, and Merrill E. Shanks (Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1961). Cloth, xi+371 pp., $4.75.

This book represents the cumulative work in the Ball State experimental program in ninth grade and exhibits a straightforward postulational approach of which ninth-grade students are capable.

After two chapters of getting the student acquainted with school, the language of sets and counting, and the symbols of arithmetic and algebra, the authors begin the study of logic with truth tables, $\alpha \rightarrow \beta$, α or β, α and β, not α, β not α, etc., which could leave the student thoroughly confused if not properly taught.

Chapters 4 and 5 develop the machinery for exhibiting the mathematics of algebra by defining the set of natural numbers $N=(1, 2, 3, 4, 5, \cdots)$, and the set of counting numbers $C=(0, 1, 2, 3, 4, 5, \cdots)$, and postulating:

1 "Existence and uniqueness: Any two counting numbers can be added and multiplied. That is, for every pair of counting numbers a and b, there is exactly one counting number, $a+b$, called the sum of a and b and exactly one counting number $a \cdot b$, called the product of a and b. Furthermore, the sum and product of two natural numbers is a natural number."

2 "Commutative Law: If a and b are counting numbers, $a+b=b+a$ and $a \cdot b=b \cdot a$."

3 "Associative Law: If a, b, and c are counting numbers, $(a+b)+c=a+(b+c)$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$."

4 "Cancellation Law: If a, b, and c are counting numbers, $a+c=b+c$, then $a=b$. If $ac=bc$ and $c \neq 0$, then $a=b$."

5 "Properties of zero and one: (1) If x is any counting number, then $x+0=x$. (2) If x is any counting number, then $x \cdot 1=x$."

6 "The Distributive Law: If a, b, and c are counting numbers, then $a \cdot (b+c)=a \cdot b+a \cdot c$."

With these postulates the authors develop precise theorems such as: "If x is any counting number, then $x+x=2x$," and "For every counting number x, $x \cdot 0=0$."

By proper definitions of subtraction, "greater than", and "less than", the authors set forth such nontrivial theorems as: "If $a>b$, then $a+c>b+c$," and "If $a>b$, and $c>d$, then $(a+c)>(b+d)$."

Having worked through the "counting numbers", postulation of the negative of a number comes naturally as $a+(-a)=0$. The integers are defined; the postulates of counting numbers are accepted for integers.

In a like manner, the rational numbers are presented and later the irrational numbers are precisely set forth.

The critics of this book would point out that there is a lack of "practical problems", but they could not say that there is a lack of exercises to strengthen the computative skills of the students; nor could they say the book fails to "cover the material" of Algebra I found in the traditional texts.

The book, to be used as a text effectively, must be used by a teacher who has a background in modern algebra and number theory and an enthusiasm for "modern" mathematics. For the adult student the book is without peer; for the "above average" ninth grader it is highly recommended; for the "average" ninth grade, worth very careful consideration.—*Berlen Flake, Dwight Eisenhower High School, Decatur, Illinois.*

Algebra I, Charles F. Brumfiel, Robert E. Eicholz, and Merrill E. Shanks (Reading, Mass.: Addition-Wesley Publishing Co., Inc., 1961). Cloth, xi+371 pp., $4.75.

Rote learning, superficial manipulative ability, and assorted artifices have long been accented in many of our first-year algebra courses.

This book, however, is an endeavor to circumvent these shortcomings through a postulational approach to the study of number systems. It is "an attempt to teach classical algebra with precision and meaning."

The purpose of the authors in this text is "to present algebra in such a way that the reasons for the different algebraic processes become evident." By presenting notions about algebraic structure and about deductive reasoning, but not by emphasizing the formal structure of them, they (Professors Brumfiel, Eicholz, and Shanks) set about the task.

Significant is the amount of intended memorization in *Algebra* I. For the entire course there are six basic assumptions: (1) existence and uniqueness, (2) commutativity, (3) associativity, (4) cancellation, (5) properties of zero and one, and (6) distributivity. Much of the other material is derived systematically.

In a large measure, this course represents an extension of arithmetic. The five sets of numbers adjoined are as follows: (1) the natural numbers and zero—the counting numbers, (2) the negative integers (preceded by a study of subtraction to show the need for new numbers), (3) zero, the natural numbers and the negatives of the natural numbers—the integers, (4) the rationals, and (5) the reals (presented as infinite decimals and fortified by postulates of order and completeness).

Unique is chapter 3 in that it introduces logical structure—not seemingly for the sake of logic, but for the sake of clarity and precision.

Noteworthy also, is chapter 11 which concerns sentences, relations, graphs, and functions; it embodies several logical and set-theoretic concepts.

Many conventional topics of first-year algebra are treated, but the stress is on skill and understanding.

The authors, in my opinion, have written a book that will elucidate many of the algebraic processes which students often question.—*E. Wayne White, Lincoln High School, Camden, Arkansas.*

Analytic Trigonometry, Paul S. Mostert (Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1960). Cloth, x+2+165 pp., $3.95.

This is a "modern" trigonometry text, using the vocabulary and concepts of set theory in Chapter I on functions. In Chapter II the sine and cosine functions are defined in terms of a terminal point function of a rotation, and the other four trigonometric functions are defined in terms of sine and cosine. These definitions begin on page 41 and terminate on page 53 of a book that contains only 120 pages plus appendices, etc. After this relatively slow start, the rest of the book is extremely brief. Chapter III consists of 12 pages on logarithms, which the author says can be covered in as little as two days if this chapter is needed. Chapter IV is an abbreviation of the customary work on the solution of triangles using four-place tables, followed by four pages on vectors. The laws of sines and cosines are derived and examples are worked with little reference to logarithms. Pages 121–144 are occupied by appendices on Polar Coordinates. Complex Numbers, Rigid Motion, Fourier Series, and Oscillations, the last three of which are not intended for class use.

Many teachers will not mourn the omission of five-place logarithm tables, the law of tangents, and the half-angle laws from the solution of triangles; but they may feel that brevity has been overdone. For instance, logarithms, "if needed", can scarcely be taught and learned thoroughly in two days. There are too few exercises to allow for variety of assignments in different sections of the class or in different years.

The book has many peculiarities of wording. On page 34, a circle is defined as a "set of points of an equal distance from a fixed point." Aside from the fact that there is no such thing as *an* equal distance, this sentence illustrates the author's difficulty with prepositions. One of several examples is on page 23, where he gives the customary formula for the distance *between* two points and calls it the distance *from* one point *to* the other.

By the definitions on pages 116 and 117 a free vector is not a vector. It is a translation, and on page 24 a translation is a movement. The definition of vector does not give to vector a direction but merely places one end point (without saying which one) at the origin. On page 19 the first sentence of the definition of multiple valued function gives both B and sets of B for the range of the function; the second sentence rules out $y=\pm\sqrt{1-x^2}$ for x between -1 and $+1$ as a multiple valued function.

There are a number of typographical errors, notable among them being the spelling of the plural of reciprocal with an apostrophe on page x of the table of contents.—*Anice Seybold, North Central College, Naperville, Illinois.*

Handbuch der Schulmathematik, Georg Wolff (ed.) (Hannover: Hermann Schroedel Verlag KG, 1960), Vol. I, *Arithmetik, Zahlenlehre.* Cloth, 295 pp., 38 German Marks.

Handbuch der Schulmathematik, Georg Wolff (ed.) (Hannover: Hermann Schroedel Verlag KG, 1961), Vol. II, *Algebra.* Cloth, 296 pp., 38 German Marks.

These are the first two volumes of a projected series of six volumes. The entire six-volume series is given the following titles: Volume I, *Arithmetic, Theory of Numbers;* Volume II, *Algebra;* Volume III, *Geometry for the Lower and Middle Grades;* Volume IV, *Geometry*

for the Upper Grades; Volume V, *Selected Topics from Geometry, Axiomatics, Philosophy, and Psychology;* Volume VI, *Analysis.*

Volume I consists of three parts: Part 1, Arithmetic of the Lower and Middle Grades; Part 2, Arithmetic of the Higher Grades; and Part 3, Arithmetic in the Industrial Associations. These titles are somewhat misleading to an American reader, for he does not find what ordinarily would be expected under such titles. The books are a curious assortment of a multitude of topics, most of which would be classified as college mathematics. The topics are developed briefly, almost in an outline fashion. Brief historical and bibliographical notes are inserted throughout the texts. Problems for the reader to work are not plentiful.

Some of the topics included in the first volume are: arithmetic and geometric progressions, differentiation, function, extrapolation and interpolation, combinations and permutations, topics from statistics such as frequency distribution, mean, normal distribution, probability, and correlation, information theory, some elements of theory of numbers including Euclid's algorithm and modulo systems, elements of set theory including a discussion of Russell's paradox and a brief discussion of the schools of formalism and intuitionism. Brief philosophical excursions accompany the presentations of many topics.

One is at a loss to suggest the most appropriate use for a text of this sort. Perhaps the most obvious place for it is a desk of a high-school teacher. He might find frequent occasions to refer to it, primarily in connection with the teaching of algebra. The knowledge of German is, of course, presupposed.

Volume II is devoted to algebra. It is also divided into three parts: Part 1, Algebra of the Middle Grades; Part 2, Algebra of the Upper Grades; and Part 3, Special Problems.

To answer the question "What is algebra?", the authors indicate that the word "algebra" is derived from the Arabic and algebra is concerned with the solution of equations. Then the authors proceed to give a formal skeleton of the theory of solution of linear equations with one unknown, followed by the application of determinants to the solution of systems of two equations with two unknowns. This is followed by a brief treatment of proportions and bookkeeping arithmetic.

Function is introduced with a sentence "The growth of a plant is a function of time." A table of values and a graph of this function follow.

A brief treatment of matrix algebra is given in connection with the study of the solution of systems of equations with several unknowns. Inequalities are introduced in connection with the topic of linear programming.

In addition to the usual algebraic methods of solving quadratic equations, ingenious geometric methods are introduced. These are used for the solution of systems of quadratics.

Algebra of the Upper Grades includes the theory of solution of third and fourth degree equations followed by a study of rational functions of the nth degree. The fundamental theorem of algebra is stated and proved. In presenting these topics, the authors assume the knowledge of trigonometry, analytic geometry, and some differential and integral calculus on the part of the readers. Part 2 of the second volume culminates in a discussion of curves given by: $x^4-y^4=-xy$, $x^4+y^4+8x^2y=0$, and $x^4+y^4-6y^3+8x^2y=0$.

Part 3 dealing with special topics includes discussions of mathematical structures, homomorphism, isomorphism, automorphism, groups, rings, linear vector spaces, linear algebras, and matrices. This is the only part of the book which can be considered to concern itself with modern abstract algebra. One is again impressed by the extreme compactness and brevity of treatment of these topics.—*Eugene D. Nichols, Florida State University, Tallahassee, Florida.*

Letters to the editor

Dear Editor:

The article "Complex Numbers and Loci," in the April, 1961, issue of THE MATHEMATICS TEACHER, neglected to treat the converse problem as all loci problems should be treated, and therefore has an incorrect statement in the sixth example, page 233.

The author, Leroy Dalton, states that the set of points in the complex plane which satisfies

$$|z-4|-|z|=2$$

is the hyperbola

$$\frac{(x-2)^2}{1}-\frac{y^2}{3}=1$$

However, only one of the two branches is in fact a solution.

Since the quantity $|z-z_0|$ represents the distance between the variable point z and a fixed point z_0, the problem could quickly be read as, "Find the locus of points each of which has the difference of the undirected distances from the points z_0 (4, 0) and z_1 (0, 0) equal to a constant, 2." Thus, the student should expect a hyperbola from the definition.

I do feel, however, that Mr. Dalton's idea is a good one.

WILLIAM G. ROUGHEAD, JR.
Northern Illinois University
DeKalb, Illinois

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Pythagorean converse

by Martin Hirsch

In the past few years, various proofs of the Pythagorean theorem have appeared in THE MATHEMATICS TEACHER. It is undoubtedly known to the reader that there are many different proofs of this well-known theorem. In the December, 1951, issue of THE MATHEMATICS TEACHER, Dr. Schaaf lists thirty articles in a bibliography of "proofs and discussions" of the Pythagorean theorem.[1] Edna Kramer, in *Main Stream of Mathematics,* writes, "Since the time of Pythagoras, one hundred or more different proofs have been given to the theorem that bears his name . . . ."[2] Strangely enough, the equally important problem of the converse of the Pythagorean theorem has been comparatively neglected. The converse states that if the square of one side of a triangle equals the sum of the squares of the other two sides, the triangle is a right triangle. The writer has been able to find only one proof of the converse. In the October, 1951, issue of THE MATHEMATICS TEACHER, we find an interesting proof by V. Thébault.[3] In it, Mr. Thébault points out that his method is independent of the Pythagorean theorem itself and uses only theorems of the first book of Euclid. In this paper I will illustrate several additional approaches to the problem of the Pythagorean converse, showing how the result can be derived—very easily in most cases—by methods of plane geometry, analytic geometry, and trigonometry.

[1] Schaaf, William L., "The Theorem of Pythagoras," THE MATHEMATICS TEACHER, XLIV (December, 1951), 585–86.

[2] Kramer, Edna E., *The Main Stream of Mathematics* (New York: Oxford University Press, 1952), p. 75.

[3] Thébault, Victor, "A Second Note on the Pythagorean Theorem," THE MATHEMATICS TEACHER, XLIV (October, 1951), 396.

PROBLEM

Given: $\triangle ABC$ and

$$a^2+b^2=c^2$$

Prove: $\triangle ABC$ is a right triangle.

METHOD I

(Refer to Fig. 1.) Let a, b, and c represent the three sides of triangle ABC, with angle C unknown. Construct a second triangle $A'B'C'$ with sides a and b including angle $C'=90°$. Let the third side equal x. Then in triangle $A'B'C'$ we have $x^2=a^2+b^2$ by the Pythagorean theorem. In triangle ABC, $c^2=a^2+b^2$ is given. Therefore $x=c$, and triangle $A'B'C'$ is congruent to triangle ABC by the SSS congruence theorem. Therefore angle C = angle $C'=90°$ since corresponding parts of congruent triangles are equal. Hence triangle ABC is a right triangle.

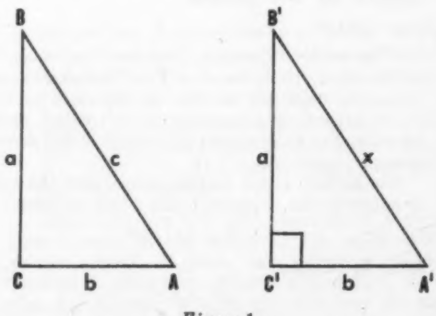

Figure 1

Method II

(Refer to Fig. 2.) Choose for the x-axis the line through AB and for the y-axis the line perpendicular to the x-axis at point A. Then the co-ordinates of A, B, and C are $A(0, 0)$, $B(x_1, 0)$ and $C(x_2, y_2)$.

Since $c^2=a^2+b^2$ we have:

$$x_1^2=x_2^2+y_2^2+(x_2-x_1)^2+y_2^2,$$

$$y_2^2=x_1x_2-x_2^2,$$

$$y_2^2=x_2(x_1-x_2),$$

$$\frac{y_2}{x_2}=\frac{-(x_2-x_1)}{y_2}$$

and

$$\frac{y_2}{x_2}=\frac{-1}{\dfrac{y_2}{x_2-x_1}}.$$

But

$$\frac{y_2}{x_2}=\text{slope of } AC=m_{AC},$$

and

$$\frac{y_2}{x_2-x_1}=\text{slope of } BC=m_{BC}.$$

Since

$$m_{AC}=\frac{-1}{m_{BC}},$$

AC is perpendicular to BC. Therefore, triangle ABC is a right triangle with the right angle at C.

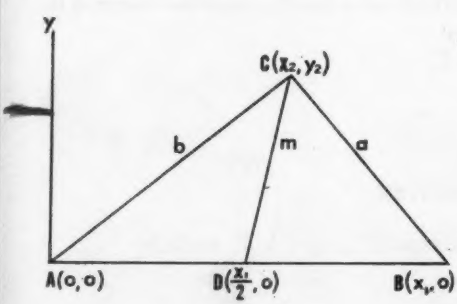

Figure 2

Method III

In Figure 2, let D, the mid-point of AB, have the co-ordinates $(x_1/2,\ 0)$. Starting with the equation $y_2^2=x_1x_2-x_2^2$ of Method II, we have $x_2^2-x_1x_2+y_2^2=0$. Multiplying each term in the above equation by 4 and adding x_1^2 to both sides of the equation gives

$$4x_2^2+x_1^2-4x_1x_2+4y_2^2=x_1^2.$$

Then

$$4\left[x_2^2+\frac{x_1^2}{4}-x_1x_2+y_2^2\right]=x_1^2,$$

$$4\left[\left(x_2-\frac{x_1}{2}\right)^2+y_2^2\right]=x_1^2,$$

and

$$2\sqrt{\left(x_2-\frac{x_1}{2}\right)^2+y_2^2}=x_1.$$

The left side of this equation equals $2CD$ and the right side equals AB in Figure 2. Hence $2CD=AB$ or $CD=AB/2$. Therefore we can circumscribe a circle around triangle ABC by taking D as center and DB as radius, thus making angle C an angle inscribed in a semicircle. Therefore angle C is a right angle and triangle ABC is a right triangle.

Method IV

We start with Heron's formula,

$$A=\sqrt{s(s-a)(s-b)(s-c)},$$

where

$$s=\frac{a+b+c}{2}.$$

In the above formula, substitute

$$\sqrt{a^2+b^2} \text{ for } c \quad\text{and}\quad \frac{a+b+c}{2} \text{ for } s.$$

We get:

$$A=\sqrt{\left[\frac{b+\sqrt{a^2+b^2}+a}{2}\right]\left[\frac{b+\sqrt{a^2+b^2}-a}{2}\right]}\cdot\sqrt{\left[\frac{a-b+\sqrt{a^2+b^2}}{2}\right]\left[\frac{a+b-\sqrt{a^2+b^2}}{2}\right]},$$

which after some simplification becomes

$$A = \sqrt{\frac{b}{2}\,[b+\sqrt{a^2+b^2}]\,\frac{b}{2}\,[-b+\sqrt{a^2+b^2}]},$$

and

$$A = \frac{b}{2}\sqrt{a^2+b^2-b^2}.$$

Thus, $A = ab/2$, which is the formula for the area of a right triangle with either a or b taken as the base and the other side as the altitude. Therefore, triangle ABC is a right triangle.

Method V

By the law of cosines, in any triangle ABC, $c^2 = a^2+b^2-2ab\cos C$. Since $a^2+b^2 = c^2$ we have $0 = -2ab\cos C$, and $\cos C = 0$. Hence, angle $C = 90°$, and triangle ABC is a right triangle.

Method VI

(Refer to Figure 3.) Draw altitude CD. Draw median CE so that $AE = EB = c/2$. Let $CD = h$ and $CE = m$. Let $DE = p$ so that $AD = c/2 - p$. In right triangle CAD,

$$b^2 = h^2 + \left(\frac{c}{2} - p\right)^2,$$

so that

$$b^2 = h^2 + \frac{c^2}{4} - cp + p^2.$$

In right triangle CBD,

$$a^2 = h^2 + \left(\frac{c}{2} + p\right)^2,$$

so that

$$a^2 = h^2 + \frac{c^2}{4} + cp + p^2.$$

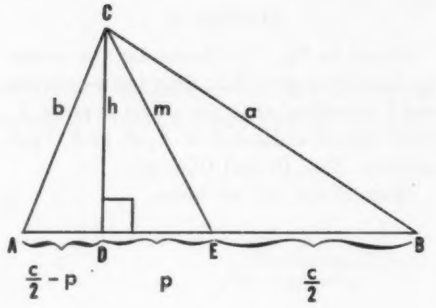

Figure 3

Adding these two equations yields $a^2+b^2 = 2h^2 + c^2/2 + 2p^2$. In this equation substitute c^2 for a^2+b^2 and multiply both sides of the equation by 2. We get $2c^2 = 4h^2 + c^2 + 4p^2$, or $c^2 = 4(h^2+p^2)$. But in right triangle CDE, $h^2+p^2 = m^2$. In $c^2 = 4(h^2+p^2)$ substitute m^2 for h^2+p^2. We get $c^2 = 4m^2$, $c = 2m$, or $m = c/2$; so that median $CE = AB/2$. Therefore we can circumscribe a circle around triangle ABC with E as center and EB as radius. Angle C is then inscribed in a semicircle and is therefore a right angle. Thus triangle ABC is a right triangle.

Methods I and VI clearly depend upon the Pythagorean theorem itself. Methods II and III make use of the distance formula which is derived from the Pythagorean theorem. Heron's theorem (Method IV) is derived in part from the Pythagorean theorem, and the law of cosines (Method V) is also derived from the Pythagorean theorem. Hence, all six of these proofs of the Pythagorean converse depend upon the Pythagorean theorem itself.

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